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SOME PARITY RESULTS FOR 16-CORES

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Abstract

L. Kolitsch and J. Sellers showed recently that $a_8(n)$, the number of 8-core partitions of n, is even when n belongs to certain arithmetic progressions. We prove a similar result for 16-cores. In doing so, we prove the surprising result that the $a_{16}(n)$, given by

$$\sum_{n\geq 0} a_{16}(n)q^n = \frac{(q^{16})_\infty^{16}}{(q)_\infty},$$

satisfy

$$a_{16}(43046721n + 457371400) \equiv a_{16}(n) \pmod 2.$$

Introduction

A partition is said to be a t-core if its Ferrers graph does not contain a hook whose length is a multiple of t. Let $a_t(n)$ denote the number of t-cores of n. Then, as F. Garvan, D. Kim and D. Stanton [1] showed,

$$\sum_{n\geq 0} a_t(n)q^n = \frac{(q^t)_\infty^t}{(q)_\infty},$$

where

$$(q)_\infty = \prod_{n\geq 1}(1-q^n).$$

By a result of Serre [7], when t is odd $a_t(n)$ satisfies congruences modulo every integer M. This follows from the finite dimensionality of spaces of holomorphic integer-weight modular forms and the theory of Hecke operators. The theory however does not predict the precise statements of the congruences. If t is even, then the generating function for $a_t(n)$ is a modular form of half-integral weight and the situation is very different. However, if one is just considering parity it is possible to retrieve the situation as follows. The classical theta function

$$\theta(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots \equiv 1 \pmod 2$$

has weight 1/2. So multiplying by $\theta(q)$ produces an integer-weight modular form which agrees with the original function modulo 2. Serre's result implies that $a_t(n) \equiv 0 \pmod 2$ for almost all n (in arithmetic density). So the existence of congruences modulo 2 might well be expected, although Serre's result does not predict the exact statement of any such congruence.

Several recent papers have been devoted to arithmetic properties of $a_t(n)$. Garvan [2] showed that when t is prime, $5 \leq t \leq 23$ and r divides $(t-1)/2$ then $a_t(n)$ satisfies certain congruences modulo r.

We [3],[4] showed that $a_4(n)$ satisfies the relations

$$a_4(9n+2) \equiv 0 \pmod 2, \quad a_4(9n+8) \equiv 0 \pmod 4,$$

and for $\lambda \geq 1,\ n \geq 0$,

$$\begin{aligned} a_4(3^{2\lambda+1}n + (5\times 3^{2\lambda}-5)/8) &= 3^\lambda a_4(3n),\\ a_4(3^{2\lambda+1}n + (13\times 3^{2\lambda}-5)/8) &= (2\times 3^\lambda - 1)a_4(3n+1),\\ a_4(3^{2\lambda+2}n + (7\times 3^{2\lambda+1}-5)/8) &= ((3^{\lambda+1}-1)/2)a_4(9n+2) \end{aligned}$$

and

$$a_4(3^{2\lambda+2}n + (23\times 3^{2\lambda+1}-5)/8) = ((3^{\lambda+1}-1)/2)a_4(9n+8)$$

and made conjectures involving $a_4(n)$ and an arbitrary odd prime p, which have since been verified by K. Ono and L. Sze [6].

L. Kolitsch and J. Sellers [5] showed that for $\lambda \geq 1,\ n \geq 0$,

$$a_8\left(81^\lambda n + \frac{7 \times 81^\lambda - 63}{24}\right) \text{ and } a_8\left(81^\lambda n + \frac{23 \times 81^\lambda - 63}{24}\right) \text{ are even.}$$

The object of this note is to establish a similar result for 16-cores, namely

Theorem. For each $\lambda \geq 1$ and all $n \geq 0$,

$$a_{16}\left(6561^\lambda n + \frac{7 \times 6561^\lambda - 255}{24}\right) \text{ and } a_{16}\left(6561^\lambda n + \frac{23 \times 6561^\lambda - 255}{24}\right) \text{ are even.}$$

Our theorem follows by induction on λ from the following results, which we shall proceed to prove.

$$a_{16}(6561n + 1903) \equiv 0 \pmod 2, \quad a_{16}(6561n + 6277) \equiv 0 \pmod 2,$$

$$a_{16}(43046721n + 12555283) \equiv 0 \pmod 2, \quad a_{16}(43046721n + 41253097) \equiv 0 \pmod 2,$$

and

$$a_{16}(43046721n + 457371400) \equiv a_{16}(n) \pmod 2.$$

Our proof employs the congruence

$$(q)_\infty^3 \equiv (q^3)_\infty + q(q^9)_\infty^3 \pmod 2,$$

which follows easily from Jacobi's identity

$$(q)_\infty^3 = \sum_{n \geq 0} (-1)^n (2n+1) q^{(n^2+n)/2}$$

together with Euler's identity

$$(q)_\infty = \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}.$$

Further, we make the following conjecture which has now been proved for $t = 4,\ 8,\ 16$.
Conjecture.

For t a power of two, $t \geq 4$, for each $\lambda \geq 1$ and all $n \geq 0$,

$$a_t\left(3^{t\lambda/2}n+\frac{7\times 3^{t\lambda/2}-(t^2-1)}{24}\right) \text{ and } a_t\left(3^{t\lambda/2}n+\frac{23\times 3^{t\lambda/2}-(t^2-1)}{24}\right) \text{ are even}$$

and

$$a_t\left(3^t n+\frac{(3^t-1)(t^2-1)}{24}\right) \equiv a_t(n) \pmod 2.$$

Indeed, it is not hard to show that our conjecture is equivalent to the following.

Let $t = 2^{k+1}$ and $c_t(n)$ be the number of representations of n in the form

$$n = 3x_0^2 + 12x_1^2 + \cdots + 3\times 4^k x_k^2$$

with the x_i odd and positive. Then

$$c_t(3^{t\lambda/2}(24n+7)),\quad c_t(3^{t\lambda/2}(24n+23)),$$
$$c_t(3^t(24n+7)) \text{ and } c_t(3^t(24n+23)) \text{ are even}$$

and

$$c_t(3^t(24n+(t^2-1))) \equiv c_t(24n+(t^2-1)) \pmod 2.$$

The proofs

We have, where all calculations are carried out modulo 2,

$$\sum_{n\geq 0} a_{16}(n)q^n = \frac{(q^{16})_\infty^{16}}{(q)_\infty} \equiv \frac{(q)_\infty^{256}}{(q)_\infty} = (q)_\infty^{255} = ((q)_\infty^3)^{85}$$
$$\equiv \left((q^3)_\infty + q(q^9)_\infty^3\right)^{85}$$
$$\equiv (q^3)_\infty^{85} + q(q^3)_\infty^{84}(q^9)_\infty^3 + q^4(q^3)_\infty^{81}(q^9)_\infty^{12} + q^5(q^3)_\infty^{80}(q^9)_\infty^{15} + q^{16}(q^3)_\infty^{69}(q^9)_\infty^{48} + q^{17}(q^3)_\infty^{68}(q^9)_\infty^{51}$$
$$+ q^{20}(q^3)_\infty^{65}(q^9)_\infty^{60} + q^{21}(q^3)_\infty^{64}(q^9)_\infty^{63} + q^{64}(q^3)_\infty^{21}(q^9)_\infty^{192} + q^{65}(q^3)_\infty^{20}(q^9)_\infty^{195} + q^{68}(q^3)_\infty^{17}(q^9)_\infty^{204}$$
$$+ q^{69}(q^3)_\infty^{16}(q^9)_\infty^{207} + q^{80}(q^3)_\infty^{5}(q^9)_\infty^{240} + q^{81}(q^3)_\infty^{4}(q^9)_\infty^{243} + q^{84}(q^3)_\infty(q^9)_\infty^{252} + q^{85}(q^9)_\infty^{255}.$$

If we now extract those terms in which the power is congruent to 1 (mod 3), divide by q and replace q^3 by q, we obtain

$$\sum_{n\geq 0} a_{16}(3n+1)q^n$$

$$
\begin{aligned}
&\equiv (q)^{84}_\infty(q^3)^3_\infty + q(q)^{81}_\infty(q^3)^{12}_\infty + q^5(q)^{69}_\infty(q^3)^{48}_\infty + q^{21}(q)^{21}_\infty(q^3)^{192}_\infty + q^{28}(q^3)^{255}_\infty \\
&\equiv (q^3)^3_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{28} + q(q^3)^{12}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{27} + q^5(q^3)^{48}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{23} \\
&\quad + q^{21}(q^3)^{192}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{7} + q^{28}(q^3)^{255}_\infty
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)^3_\infty\Big((q^3)^{28}_\infty + q^4(q^3)^{24}_\infty(q^9)^{12}_\infty + q^8(q^3)^{20}_\infty(q^9)^{24}_\infty + q^{12}(q^3)^{16}(q^9)^{36}_\infty + q^{16}(q^3)^{12}_\infty(q^9)^{48}_\infty \\
&\qquad\qquad + q^{20}(q^3)^{8}_\infty(q^9)^{60}_\infty + q^{24}(q^3)^{4}_\infty(q^9)^{72}_\infty + q^{28}(q^9)^{84}_\infty\Big) \\
&\quad + q(q^3)^{12}_\infty\Big((q^3)^{27}_\infty + q(q^3)^{26}_\infty(q^9)^{3}_\infty + q^2(q^3)^{25}_\infty(q^9)^{6}_\infty + q^3(q^3)^{24}_\infty(q^9)^{9}_\infty + q^8(q^3)^{19}_\infty(q^9)^{24}_\infty \\
&\qquad\qquad + q^9(q^3)^{18}_\infty(q^9)^{27}_\infty + q^{10}(q^3)^{17}_\infty(q^9)^{30}_\infty + q^{11}(q^3)^{16}_\infty(q^9)^{33}_\infty + q^{16}(q^3)^{11}_\infty(q^9)^{48}_\infty \\
&\qquad\qquad + q^{17}(q^3)^{10}_\infty(q^9)^{51}_\infty + q^{18}(q^3)^{9}_\infty(q^9)^{54}_\infty + q^{19}(q^3)^{8}_\infty(q^9)^{57}_\infty + q^{24}(q^3)^{3}_\infty(q^9)^{72}_\infty \\
&\qquad\qquad + q^{25}(q^3)^{2}_\infty(q^9)^{75}_\infty + q^{26}(q^3)_\infty(q^9)^{78}_\infty + q^{27}(q^9)^{81}_\infty\Big) \\
&\quad + q^5(q^3)^{48}_\infty\Big((q^3)^{23}_\infty + q(q^3)^{22}_\infty(q^9)^{3}_\infty + q^2(q^3)^{21}_\infty(q^9)^{6}_\infty + q^3(q^3)^{20}_\infty(q^9)^{9}_\infty + q^4(q^3)^{19}_\infty(q^9)^{12}_\infty \\
&\qquad\qquad + q^5(q^3)^{18}_\infty(q^9)^{15}_\infty + q^6(q^3)^{17}_\infty(q^9)^{18}_\infty + q^7(q^3)^{16}_\infty(q^9)^{21}_\infty + q^{16}(q^3)^{7}_\infty(q^9)^{48}_\infty \\
&\qquad\qquad + q^{17}(q^3)^{6}_\infty(q^9)^{51}_\infty + q^{18}(q^3)^{5}_\infty(q^9)^{54}_\infty + q^{19}(q^3)^{4}_\infty(q^9)^{57}_\infty + q^{20}(q^3)^{3}_\infty(q^9)^{60}_\infty \\
&\qquad\qquad + q^{21}(q^3)^{2}_\infty(q^9)^{63}_\infty + q^{22}(q^3)_\infty(q^9)^{66}_\infty + q^{23}(q^9)^{69}_\infty\Big) \\
&\quad + q^{21}(q^3)^{192}_\infty\Big((q^3)^{7}_\infty + q(q^3)^{6}_\infty(q^9)^{3}_\infty + q^2(q^3)^{5}_\infty(q^9)^{6}_\infty + q^3(q^3)^{4}_\infty(q^9)^{9}_\infty + q^4(q^3)^{3}_\infty(q^9)^{12}_\infty \\
&\qquad\qquad + q^5(q^3)^{2}_\infty(q^9)^{15}_\infty + q^6(q^3)_\infty(q^9)^{18}_\infty + q^7(q^9)^{21}_\infty\Big) \\
&\quad + q^{28}(q^3)^{255}
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)^{31}_\infty + q(q^3)^{39}_\infty + q^2(q^3)^{38}_\infty(q^9)^{3}_\infty + q^3(q^3)^{37}_\infty(q^9)^{6}_\infty + q^4(q^3)^{27}_\infty(q^9)^{12}_\infty + q^4(q^3)^{36}_\infty(q^9)^{9}_\infty \\
&\quad + q^5(q^3)^{71}_\infty + q^6(q^3)^{70}_\infty(q^9)^{3}_\infty + q^7(q^3)^{69}_\infty(q^9)^{6}_\infty + q^8(q^3)^{23}_\infty(q^9)^{24}_\infty + q^8(q^3)^{68}_\infty(q^9)^{9}_\infty \\
&\quad + q^9(q^3)^{31}_\infty(q^9)^{24}_\infty + q^9(q^3)^{67}_\infty(q^9)^{12}_\infty + q^{10}(q^3)^{30}_\infty(q^9)^{27}_\infty + q^{10}(q^3)^{66}_\infty(q^9)^{15}_\infty + q^{11}(q^3)^{29}_\infty(q^9)^{30}_\infty \\
&\quad + q^{11}(q^3)^{65}_\infty(q^9)^{18}_\infty + q^{12}(q^3)^{19}_\infty(q^9)^{36}_\infty + q^{12}(q^3)^{28}_\infty(q^9)^{33}_\infty + q^{12}(q^3)^{64}_\infty(q^9)^{21}_\infty + q^{16}(q^3)^{15}_\infty(q^9)^{48}_\infty \\
&\quad + q^{17}(q^3)^{23}_\infty(q^9)^{48}_\infty + q^{18}(q^3)^{22}_\infty(q^9)^{51}_\infty + q^{19}(q^3)^{21}_\infty(q^9)^{54}_\infty + q^{20}(q^3)^{11}_\infty(q^9)^{60}_\infty + q^{20}(q^3)^{20}_\infty(q^9)^{57}_\infty \\
&\quad + q^{21}(q^3)^{55}_\infty(q^9)^{48}_\infty + q^{21}(q^3)^{199}_\infty + q^{22}(q^3)^{54}_\infty(q^9)^{51}_\infty + q^{22}(q^3)^{198}_\infty(q^9)^{3}_\infty + q^{23}(q^3)^{53}_\infty(q^9)^{54}_\infty \\
&\quad + q^{23}(q^3)^{197}_\infty(q^9)^{6}_\infty + q^{24}(q^3)^{7}_\infty(q^9)^{72}_\infty + q^{24}(q^3)^{52}_\infty(q^9)^{57}_\infty + q^{24}(q^3)^{196}_\infty(q^9)^{9}_\infty + q^{25}(q^3)^{15}_\infty(q^9)^{72}_\infty \\
&\quad + q^{25}(q^3)^{51}_\infty(q^9)^{60}_\infty + q^{25}(q^3)^{195}_\infty(q^9)^{12}_\infty + q^{26}(q^3)^{14}_\infty(q^9)^{75}_\infty + q^{26}(q^3)^{50}_\infty(q^9)^{63}_\infty + q^{26}(q^3)^{194}_\infty(q^9)^{15}_\infty \\
&\quad + q^{27}(q^3)^{13}_\infty(q^9)^{78}_\infty + q^{27}(q^3)^{49}_\infty(q^9)^{66}_\infty + q^{27}(q^3)^{193}_\infty(q^9)^{18}_\infty + q^{28}(q^3)^{3}_\infty(q^9)^{84}_\infty + q^{28}(q^3)^{12}_\infty(q^9)^{81}_\infty
\end{aligned}
$$

$$+ q^{28}(q^3)_\infty^{48}(q^9)_\infty^{69} + q^{28}(q^3)_\infty^{192}(q^9)_\infty^{21} + q^{28}(q^3)_\infty^{255}.$$

If we extract those terms in which the power is congruent to 1 (mod 3), divide by q and replace q^3 by q, we find

$$\sum_{n\ge 0} a_{16}(9n+4)q^n$$
$$\equiv (q)_\infty^{39} + q(q)_\infty^{27}(q^3)_\infty^{12} + q(q)_\infty^{36}(q^3)_\infty^{9} + q^2(q)_\infty^{69}(q^3)_\infty^{6} + q^3(q)_\infty^{30}(q^3)_\infty^{27} + q^3(q)_\infty^{66}(q^3)_\infty^{15}$$
$$+ q^5(q)_\infty^{15}(q^3)_\infty^{48} + q^6(q)_\infty^{21}(q^3)_\infty^{54} + q^7(q)_\infty^{54}(q^3)_\infty^{51} + q^7(q)_\infty^{198}(q^3)_\infty^{3} + q^8(q)_\infty^{15}(q^3)_\infty^{72}$$
$$+ q^8(q)_\infty^{51}(q^3)_\infty^{60} + q^8(q)_\infty^{195}(q^3)_\infty^{12} + q^9(q)_\infty^{3}(q^3)_\infty^{84} + q^9(q)_\infty^{12}(q^3)_\infty^{81} + q^9(q)_\infty^{48}(q^3)_\infty^{69}$$
$$+ q^9(q)_\infty^{192}(q^3)_\infty^{21} + q^9(q)_\infty^{255}$$

$$\equiv \left((q^3)_\infty + q(q^9)_\infty^3\right)^{13} + q(q^3)_\infty^{12}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{9} + q(q^3)_\infty^{9}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{12}$$
$$+ q^2(q^3)_\infty^{6}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{23} + q^3(q^3)_\infty^{27}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{10} + q^3(q^3)_\infty^{15}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{22}$$
$$+ q^5(q^3)_\infty^{48}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{5} + q^6(q^3)_\infty^{54}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{7} + q^7(q^3)_\infty^{51}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{18}$$
$$+ q^7(q^3)_\infty^{3}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{66} + q^8(q^3)_\infty^{72}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{5} + q^8(q^3)_\infty^{60}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{17}$$
$$+ q^8(q^3)_\infty^{12}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{65} + q^9(q^3)_\infty^{84}\left((q^3)_\infty + q(q^9)_\infty^3\right) + q^9(q^3)_\infty^{81}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{4}$$
$$+ q^9(q^3)_\infty^{69}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{16} + q^9(q^3)_\infty^{21}\left((q^3)_\infty + q(q^9)_\infty^3\right)^{64} + q^9\left((q^3)_\infty + q(q^9)_\infty^3\right)^{85}$$

$$\equiv \big((q^3)_\infty^{13} + q(q^3)_\infty^{12}(q^9)_\infty^{3} + q^4(q^3)_\infty^{9}(q^9)_\infty^{12} + q^5(q^3)_\infty^{8}(q^9)_\infty^{15} + q^8(q^3)_\infty^{5}(q^9)_\infty^{24} + q^9(q^3)_\infty^{4}(q^9)_\infty^{27}$$
$$+q^{12}(q^3)_\infty(q^9)_\infty^{36} + q^{13}(q^9)_\infty^{39}\big)$$
$$+ q(q^3)_\infty^{12}\left((q^3)_\infty^{9} + q(q^3)_\infty^{8}(q^9)_\infty^{3} + q^8(q^3)_\infty(q^9)_\infty^{24} + q^9(q^9)_\infty^{27}\right)$$
$$+ q(q^3)_\infty^{9}\left((q^3)_\infty^{12} + q^4(q^3)_\infty^{8}(q^9)_\infty^{12} + q^8(q^3)_\infty^{4}(q^9)_\infty^{24} + q^{12}(q^9)_\infty^{36}\right)$$
$$+ q^2(q^3)_\infty^{6}\big((q^3)_\infty^{23} + q(q^3)_\infty^{22}(q^9)_\infty^{3} + q^2(q^3)_\infty^{21}(q^9)_\infty^{6} + q^3(q^3)_\infty^{20}(q^9)_\infty^{9} + q^4(q^3)_\infty^{19}(q^9)_\infty^{12}$$
$$+q^5(q^3)_\infty^{18}(q^9)_\infty^{15} + q^6(q^3)_\infty^{17}(q^9)_\infty^{18} + q^7(q^3)_\infty^{16}(q^9)_\infty^{21} + q^{16}(q^3)_\infty^{7}(q^9)_\infty^{48}$$
$$+q^{17}(q^3)_\infty^{6}(q^9)_\infty^{51} + q^{18}(q^3)_\infty^{5}(q^9)_\infty^{54} + q^{19}(q^3)_\infty^{4}(q^9)_\infty^{57} + q^{20}(q^3)_\infty^{3}(q^9)_\infty^{60}$$
$$+q^{21}(q^3)_\infty^{2}(q^9)_\infty^{63} + q^{22}(q^3)_\infty(q^9)_\infty^{66} + q^{23}(q^9)_\infty^{69}\big)$$
$$+ q^3(q^3)_\infty^{27}\left((q^3)_\infty^{10} + q^2(q^3)_\infty^{8}(q^9)_\infty^{6} + q^8(q^3)_\infty^{2}(q^9)_\infty^{24} + q^{10}(q^9)_\infty^{30}\right)$$
$$+ q^3(q^3)_\infty^{15}\big((q^3)_\infty^{22} + q^2(q^3)_\infty^{20}(q^9)_\infty^{6} + q^4(q^3)_\infty^{18}(q^9)_\infty^{12} + q^6(q^3)_\infty^{16}(q^9)_\infty^{18} + q^{16}(q^3)_\infty^{6}(q^9)_\infty^{48}$$

$$
\begin{aligned}
&\quad +q^{18}(q^3)^4_\infty(q^9)^{54}_\infty+q^{20}(q^3)^2_\infty(q^9)^{60}_\infty+q^{22}(q^9)^{66}_\infty\big)\\
&+q^5(q^3)^{48}_\infty\big((q^3)^5_\infty+q(q^3)^4_\infty(q^9)^3_\infty+q^4(q^3)_\infty(q^9)^{12}_\infty+q^5(q^9)^{15}_\infty\big)\\
&+q^6(q^3)^{54}_\infty\big((q^3)^7_\infty+q(q^3)^6_\infty(q^9)^3_\infty+q^2(q^3)^5_\infty(q^9)^6_\infty+q^3(q^3)^4_\infty(q^9)^9_\infty+q^4(q^3)^3_\infty(q^9)^{12}_\infty\\
&\quad +q^5(q^3)^2_\infty(q^9)^{15}_\infty+q^6(q^3)_\infty(q^9)^{18}_\infty+q^7(q^9)^{21}_\infty\big)\\
&+q^7(q^3)^{51}_\infty\big((q^3)^{18}_\infty+q^2(q^3)^{16}_\infty(q^9)^6_\infty+q^{16}(q^3)^2_\infty(q^9)^{48}_\infty+q^{18}(q^9)^{54}_\infty\big)\\
&+q^7(q^3)^3_\infty\big((q^3)^{66}_\infty+q^2(q^3)^{64}_\infty(q^9)^6_\infty+q^{64}(q^3)^2_\infty(q^9)^{192}_\infty+q^{66}(q^9)^{198}_\infty\big)\\
&+q^8(q^3)^{72}_\infty\big((q^3)^5_\infty+q(q^3)^4_\infty(q^9)^3_\infty+q^4(q^3)_\infty(q^9)^{12}_\infty+q^5(q^9)^{15}_\infty\big)\\
&+q^8(q^3)^{60}_\infty\big((q^3)^{17}_\infty+q(q^3)^{16}_\infty(q^9)^3_\infty+q^{16}(q^3)_\infty(q^9)^{48}_\infty+q^{17}(q^9)^{51}_\infty\big)\\
&+q^8(q^3)^{12}_\infty\big((q^3)^{65}_\infty+q(q^3)^{64}_\infty(q^9)^3_\infty+q^{64}(q^3)_\infty(q^9)^{192}_\infty+q^{65}(q^9)^{195}_\infty\big)\\
&+q^9(q^3)^{84}_\infty\big((q^3)_\infty+q(q^9)^3_\infty\big)\\
&+q^9(q^3)^{81}_\infty\big((q^3)^4_\infty+q^4(q^9)^{12}_\infty\big)\\
&+q^9(q^3)^{69}_\infty\big((q^3)^{16}_\infty+q^{16}(q^9)^{48}_\infty\big)\\
&+q^9(q^3)^{21}_\infty\big((q^3)^{64}_\infty+q^{64}(q^9)^{192}_\infty\big)\\
&+q^9\big((q^3)^{85}_\infty+q(q^3)^{84}_\infty(q^9)^3_\infty+q^4(q^3)^{81}_\infty(q^9)^{12}_\infty+q^5(q^3)^{80}_\infty(q^9)^{15}_\infty+q^{16}(q^3)^{69}_\infty(q^9)^{48}_\infty\\
&\quad +q^{17}(q^3)^{68}_\infty(q^9)^{51}_\infty+q^{20}(q^3)^{65}_\infty(q^9)^{60}_\infty+q^{21}(q^3)^{64}_\infty(q^9)^{63}_\infty+q^{64}(q^3)^{21}_\infty(q^9)^{192}_\infty\\
&\quad +q^{65}(q^3)^{20}_\infty(q^9)^{195}_\infty+q^{68}(q^3)^{17}_\infty(q^9)^{204}_\infty+q^{69}(q^3)^{16}_\infty(q^9)^{207}_\infty+q^{80}(q^3)^5_\infty(q^9)^{240}_\infty\\
&\quad +q^{81}(q^3)^4_\infty(q^9)^{243}_\infty+q^{84}(q^3)_\infty(q^9)^{252}_\infty+q^{85}(q^9)^{255}_\infty\big)
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)^{13}_\infty+q(q^3)^{12}_\infty(q^9)^3_\infty+q^2(q^3)^{20}_\infty(q^9)^3_\infty+q^2(q^3)^{29}_\infty+q^3(q^3)^{28}_\infty(q^9)^3_\infty+q^4(q^3)^9_\infty(q^9)^{12}_\infty\\
&+q^4(q^3)^{27}_\infty(q^9)^6_\infty+q^5(q^3)^8_\infty(q^9)^{15}_\infty+q^5(q^3)^{17}_\infty(q^9)^{12}_\infty+q^5(q^3)^{26}_\infty(q^9)^9_\infty+q^5(q^3)^{53}_\infty\\
&+q^6(q^3)^{25}_\infty(q^9)^{12}_\infty+q^6(q^3)^{52}_\infty(q^9)^3_\infty+q^6(q^3)^{61}_\infty+q^7(q^3)^{24}_\infty(q^9)^{15}_\infty+q^7(q^3)^{33}_\infty(q^9)^{12}_\infty\\
&+q^7(q^3)^{60}_\infty(q^9)^3_\infty+q^8(q^3)^5_\infty(q^9)^{24}_\infty+q^8(q^3)^{23}_\infty(q^9)^{18}_\infty+q^8(q^3)^{59}_\infty(q^9)^6_\infty+q^8(q^3)^{77}_\infty\\
&+q^9(q^3)^4_\infty(q^9)^{27}_\infty+q^9(q^3)^{22}_\infty(q^9)^{21}_\infty+q^9(q^3)^{31}_\infty(q^9)^{18}_\infty+q^9(q^3)^{49}_\infty(q^9)^{12}_\infty+q^9(q^3)^{58}_\infty(q^9)^9_\infty\\
&+q^9(q^3)^{76}_\infty(q^9)^3_\infty+q^9(q^3)^{85}_\infty+q^{10}(q^3)^{12}_\infty(q^9)^{27}_\infty+q^{10}(q^3)^{48}_\infty(q^9)^{15}_\infty+q^{10}(q^3)^{57}_\infty(q^9)^{12}_\infty\\
&+q^{11}(q^3)^{29}_\infty(q^9)^{24}_\infty+q^{11}(q^3)^{56}_\infty(q^9)^{15}_\infty+q^{12}(q^3)_\infty(q^9)^{36}_\infty+q^{12}(q^3)^{55}_\infty(q^9)^{18}_\infty+q^{12}(q^3)^{73}_\infty(q^9)^{12}_\infty\\
&+q^{13}(q^9)^{39}_\infty+q^{13}(q^3)^9_\infty(q^9)^{36}_\infty+q^{13}(q^3)^{27}_\infty(q^9)^{30}_\infty+q^{13}(q^3)^{54}_\infty(q^9)^{21}_\infty+q^{13}(q^3)^{72}_\infty(q^9)^{15}_\infty\\
&+q^{14}(q^3)^{80}_\infty(q^9)^{15}_\infty+q^{18}(q^3)^{13}_\infty(q^9)^{48}_\infty+q^{19}(q^3)^{12}_\infty(q^9)^{51}_\infty+q^{19}(q^3)^{21}_\infty(q^9)^{48}_\infty+q^{20}(q^3)^{11}_\infty(q^9)^{54}_\infty
\end{aligned}
$$

$$+ q^{21}(q^3)^{10}_\infty(q^9)^{57}_\infty + q^{21}(q^3)^{19}_\infty(q^9)^{54}_\infty + q^{22}(q^3)^{9}_\infty(q^9)^{60}_\infty + q^{23}(q^3)^{8}_\infty(q^9)^{63}_\infty + q^{23}(q^3)^{17}_\infty(q^9)^{60}_\infty$$
$$+ q^{23}(q^3)^{53}_\infty(q^9)^{48}_\infty + q^{24}(q^3)^{7}_\infty(q^9)^{66}_\infty + q^{24}(q^3)^{61}_\infty(q^9)^{48}_\infty + q^{25}(q^3)^{6}_\infty(q^9)^{69}_\infty + q^{25}(q^3)^{15}_\infty(q^9)^{66}_\infty$$
$$+ q^{25}(q^3)^{51}_\infty(q^9)^{54}_\infty + q^{25}(q^3)^{60}_\infty(q^9)^{51}_\infty + q^{26}(q^3)^{68}_\infty(q^9)^{51}_\infty + q^{29}(q^3)^{65}_\infty(q^9)^{60}_\infty + q^{30}(q^3)^{64}_\infty(q^9)^{63}_\infty$$
$$+ q^{71}(q^3)^{5}_\infty(q^9)^{192}_\infty + q^{72}(q^3)^{13}_\infty(q^9)^{192}_\infty + q^{73}(q^3)^{3}_\infty(q^9)^{198}_\infty + q^{73}(q^3)^{12}_\infty(q^9)^{195}_\infty + q^{74}(q^3)^{20}_\infty(q^9)^{195}_\infty$$
$$+ q^{77}(q^3)^{17}_\infty(q^9)^{204}_\infty + q^{78}(q^3)^{16}_\infty(q^9)^{207}_\infty + q^{89}(q^3)^{5}_\infty(q^9)^{240}_\infty + q^{90}(q^3)^{4}_\infty(q^9)^{243}_\infty + q^{93}(q^3)_\infty(q^9)^{252}_\infty$$
$$+ q^{94}(q^9)^{255}_\infty.$$

If once again we extract those terms in which the power is congruent to 1 (mod 3), divide by q and replace q^3 by q, we obtain

$$\sum_{n\geq 0} a_{16}(27n+13)q^n$$
$$\equiv (q)^{12}_\infty(q^3)^{3}_\infty + q(q)^{9}_\infty(q^3)^{12}_\infty + q(q)^{27}_\infty(q^3)^{6}_\infty + q^2(q)^{24}_\infty(q^3)^{15}_\infty + q^2(q)^{33}_\infty(q^3)^{12}_\infty + q^2(q)^{60}_\infty(q^3)^{3}_\infty$$
$$+ q^3(q)^{12}_\infty(q^3)^{27}_\infty + q^3(q)^{48}_\infty(q^3)^{15}_\infty + q^3(q)^{57}_\infty(q^3)^{12}_\infty + q^4(q^3)^{39}_\infty + q^4(q)^{9}_\infty(q^3)^{36}_\infty + q^4(q)^{27}_\infty(q^3)^{30}_\infty$$
$$+ q^4(q)^{54}_\infty(q^3)^{21}_\infty + q^4(q)^{72}_\infty(q^3)^{15}_\infty + q^6(q)^{12}_\infty(q^3)^{51}_\infty + q^6(q)^{21}_\infty(q^3)^{48}_\infty + q^7(q)^{9}_\infty(q^3)^{60}_\infty + q^8(q)^{6}_\infty(q^3)^{69}_\infty$$
$$+ q^8(q)^{15}_\infty(q^3)^{66}_\infty + q^8(q)^{51}_\infty(q^3)^{54}_\infty + q^8(q)^{60}_\infty(q^3)^{51}_\infty + q^{24}(q)^{3}_\infty(q^3)^{198}_\infty + q^{24}(q)^{12}_\infty(q^3)^{195}_\infty + q^{31}(q^3)^{255}_\infty$$

$$\equiv (q^3)^3_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^4 + q(q^3)^{12}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^3 + q(q^3)^{6}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^9$$
$$+ q^2(q^3)^{15}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^8 + q^2(q^3)^{12}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{11} + q^2(q^3)^{3}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{20}$$
$$+ q^3(q^3)^{27}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^4 + q^3(q^3)^{15}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{16} + q^3(q^3)^{12}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{19}$$
$$+ q^4(q^3)^{39}_\infty + q^4(q^3)^{36}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^3 + q^4(q^3)^{30}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^9$$
$$+ q^4(q^3)^{21}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{18} + q^4(q^3)^{15}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{24} + q^6(q^3)^{51}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^4$$
$$+ q^6(q^3)^{48}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^7 + q^7(q^3)^{60}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^3 + q^8(q^3)^{69}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^2$$
$$+ q^8(q^3)^{66}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^5 + q^8(q^3)^{54}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{17} + q^8(q^3)^{51}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^{20}$$
$$+ q^{24}(q^3)^{198}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right) + q^{24}(q^3)^{195}_\infty\left((q^3)_\infty + q(q^9)^3_\infty\right)^4 + q^{31}(q^3)^{255}_\infty$$

$$\equiv (q^3)^3_\infty\left((q^3)^4_\infty + q^4(q^9)^{12}_\infty\right)$$
$$+ q(q^3)^{12}_\infty\left((q^3)^3_\infty + q(q^3)^2_\infty(q^9)^3_\infty + q^2(q^3)_\infty(q^9)^6_\infty + q^3(q^9)^9_\infty\right)$$

$$
\begin{aligned}
&+ q(q^3)_\infty^6 \left((q^3)_\infty^9 + q(q^3)_\infty^8(q^9)_\infty^3 + q^8(q^3)_\infty(q^9)_\infty^{24} + q^9(q^9)_\infty^{27}\right)\\
&+ q^2(q^3)_\infty^{15} \left((q^3)_\infty^8 + q^8(q^9)_\infty^{24}\right)\\
&+ q^2(q^3)_\infty^{12} \left((q^3)_\infty^{11} + q(q^3)_\infty^{10}(q^9)_\infty^3 + q^2(q^3)_\infty^9(q^9)_\infty^6 + q^3(q^3)_\infty^8(q^9)_\infty^9 + q^8(q^3)_\infty^3(q^9)_\infty^{24}\right.\\
&\qquad\quad \left.+q^9(q^3)_\infty^2(q^9)_\infty^{27} + q^{10}(q^3)_\infty(q^9)_\infty^{30} + q^{11}(q^9)_\infty^{33}\right)\\
&+ q^2(q^3)_\infty^3 \left((q^3)_\infty^{20} + q^4(q^3)_\infty^{16}(q^9)_\infty^{12} + q^{16}(q^3)_\infty^4(q^9)_\infty^{48} + q^{20}(q^9)_\infty^{60}\right)\\
&+ q^3(q^3)_\infty^{27} \left((q^3)_\infty^4 + q^4(q^9)_\infty^{12}\right)\\
&+ q^3(q^3)_\infty^{15} \left((q^3)_\infty^{16} + q^{16}(q^9)_\infty^{48}\right)\\
&+ q^3(q^3)_\infty^{12} \left((q^3)_\infty^{19} + q(q^3)_\infty^{18}(q^9)_\infty^3 + q^2(q^3)_\infty^{17}(q^9)_\infty^6 + q^3(q^3)_\infty^{16}(q^9)_\infty^9 + q^{16}(q^3)_\infty^3(q^9)_\infty^{48}\right.\\
&\qquad\quad \left.+q^{17}(q^3)_\infty^2(q^9)_\infty^{51} + q^{18}(q^3)_\infty(q^9)_\infty^{54} + q^{19}(q^9)_\infty^{57}\right)\\
&+ q^4(q^3)_\infty^{39}\\
&+ q^4(q^3)_\infty^{36} \left((q^3)_\infty^3 + q(q^3)_\infty^2(q^9)_\infty^3 + q^2(q^3)_\infty(q^9)_\infty^6 + q^3(q^9)_\infty^9\right)\\
&+ q^4(q^3)_\infty^{30} \left((q^3)_\infty^9 + q(q^3)_\infty^8(q^9)_\infty^3 + q^8(q^3)_\infty(q^9)_\infty^{24} + q^9(q^9)_\infty^{27}\right)\\
&+ q^4(q^3)_\infty^{21} \left((q^3)^{18} + q^2(q^3)_\infty^{16}(q^9)_\infty^6 + q^{16}(q^3)_\infty^2(q^9)_\infty^{48} + q^{18}(q^9)_\infty^{54}\right)\\
&+ q^4(q^3)_\infty^{15} \left((q^3)_\infty^{24} + q^8(q^3)_\infty^{16}(q^9)_\infty^{24} + q^{16}(q^3)_\infty^8(q^9)_\infty^{48} + q^{24}(q^9)_\infty^{72}\right)\\
&+ q^6(q^3)_\infty^{51} \left((q^3)_\infty^4 + q^4(q^9)_\infty^{12}\right)\\
&+ q^6(q^3)_\infty^{48} \left((q^3)_\infty^7 + q(q^3)_\infty^6(q^9)_\infty^3 + q^2(q^3)_\infty^5(q^9)_\infty^6 + q^3(q^3)_\infty^4(q^9)_\infty^9 + q^4(q^3)_\infty^3(q^9)_\infty^{12}\right.\\
&\qquad\quad \left.+q^5(q^3)_\infty^2(q^9)_\infty^{15} + q^6(q^3)_\infty(q^9)_\infty^{18} + q^7(q^9)_\infty^{21}\right)\\
&+ q^7(q^3)_\infty^{60} \left((q^3)_\infty^3 + q(q^3)_\infty^2(q^9)_\infty^3 + q^2(q^3)_\infty(q^9)_\infty^6 + q^3(q^9)_\infty^9\right)\\
&+ q^8(q^3)_\infty^{69} \left((q^3)_\infty^2 + q^2(q^9)_\infty^6\right)\\
&+ q^8(q^3)_\infty^{66} \left((q^3)_\infty^5 + q(q^3)_\infty^4(q^9)_\infty^3 + q^4(q^3)_\infty(q^9)_\infty^{12} + q^5(q^9)_\infty^{15}\right)\\
&+ q^8(q^3)_\infty^{54} \left((q^3)_\infty^{17} + q(q^3)_\infty^{16}(q^9)_\infty^3 + q^{16}(q^3)_\infty(q^9)_\infty^{48} + q^{17}(q^9)_\infty^{51}\right)\\
&+ q^8(q^3)_\infty^{51} \left((q^3)_\infty^{20} + q^4(q^3)_\infty^{16}(q^9)_\infty^{12} + q^{16}(q^3)_\infty^4(q^9)_\infty^{48} + q^{20}(q^9)_\infty^{60}\right)\\
&+ q^{24}(q^3)_\infty^{198} \left((q^3)_\infty + q(q^9)_\infty^3\right)\\
&+ q^{24}(q^3)_\infty^{195} \left((q^3)_\infty^4 + q^4(q^9)_\infty^{12}\right) + q^{31}(q^3)_\infty^{255}
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)_\infty^7 + q^2(q^3)_\infty^{23} + q^3(q^3)_\infty^{13}(q^9)_\infty^6 + q^3(q^3)_\infty^{22}(q^9)_\infty^3 + q^3(q^3)_\infty^{31} + q^4(q^3)_\infty^3(q^9)_\infty^{12}\\
&+ q^4(q^3)_\infty^{12}(q^9)_\infty^9 + q^4(q^3)_\infty^{21}(q^9)_\infty^6 + q^4(q^3)_\infty^{30}(q^9)_\infty^3 + q^4(q^3)_\infty^{39} + q^5(q^3)_\infty^{20}(q^9)_\infty^9
\end{aligned}
$$

$$+ q^5(q^3)^{29}_\infty(q^9)^6_\infty + q^6(q^3)^{19}_\infty(q^9)^{12}_\infty + q^6(q^3)^{28}_\infty(q^9)^9_\infty + q^7(q^3)^{27}_\infty(q^9)^{12}_\infty + q^7(q^3)^{36}_\infty(q^9)^9_\infty$$
$$+ q^7(q^3)^{54}_\infty(q^9)^3_\infty + q^7(q^3)^{63}_\infty + q^8(q^3)^{53}_\infty(q^9)^6_\infty + q^8(q^3)^{62}_\infty(q^9)^3_\infty + q^9(q^3)^7_\infty(q^9)^{24}_\infty$$
$$+ q^9(q^3)^{52}_\infty(q^9)^9_\infty + q^9(q^3)^{61}_\infty(q^9)^6_\infty + q^{10}(q^3)^6_\infty(q^9)^{27}_\infty + q^{10}(q^3)^{60}_\infty(q^9)^9_\infty + q^{10}(q^3)^{69}_\infty(q^9)^6_\infty$$
$$+ q^{11}(q^3)^{14}_\infty(q^9)^{27}_\infty + q^{11}(q^3)^{50}_\infty(q^9)^{15}_\infty + q^{12}(q^3)^{13}_\infty(q^9)^{30}_\infty + q^{12}(q^3)^{49}_\infty(q^9)^{18}_\infty + q^{13}(q^3)^{12}_\infty(q^9)^{33}_\infty$$
$$+ q^{13}(q^3)^{30}_\infty(q^9)^{27}_\infty + q^{13}(q^3)^{48}_\infty(q^9)^{21}_\infty + q^{13}(q^3)^{66}_\infty(q^9)^{15}_\infty + q^{18}(q^3)^7_\infty(q^9)^{48}_\infty + q^{20}(q^3)^{14}_\infty(q^9)^{51}_\infty$$
$$+ q^{21}(q^3)^{13}_\infty(q^9)^{54}_\infty + q^{22}(q^3)^3_\infty(q^9)^{60}_\infty + q^{22}(q^3)^{12}_\infty(q^9)^{57}_\infty + q^{22}(q^3)^{21}_\infty(q^9)^{54}_\infty + q^{25}(q^3)^{54}_\infty(q^9)^{51}_\infty$$
$$+ q^{25}(q^3)^{198}_\infty(q^9)^3_\infty + q^{28}(q^3)^{15}_\infty(q^9)^{72}_\infty + q^{28}(q^3)^{51}_\infty(q^9)^{60}_\infty + q^{28}(q^3)^{195}_\infty(q^9)^{12}_\infty + q^{31}(q^3)^{255}_\infty,$$

If yet again we extract those terms in which the power of q is $1 \pmod 3$, divide by q and replace q^3 by q, we find

$$\sum_{n\geq 0} a_{16}(81n+40)q^n$$
$$\equiv q(q)^3_\infty(q^3)^{12}_\infty + q(q)^{12}_\infty(q^3)^9_\infty + q(q)^{21}_\infty(q^3)^6_\infty + q(q)^{30}_\infty(q^3)^3_\infty + q(q)^{39}_\infty + q^2(q)^{27}_\infty(q^3)^{12}_\infty$$
$$+ q^2(q)^{36}_\infty(q^3)^9_\infty + q^2(q)^{54}_\infty(q^3)^3_\infty + q^2(q)^{63}_\infty + q^3(q)^6_\infty(q^3)^{27}_\infty + q^3(q)^{60}_\infty(q^3)^9_\infty + q^3(q)^{69}_\infty(q^3)^6_\infty$$
$$+ q^4(q)^{12}_\infty(q^3)^{33}_\infty + q^4(q)^{30}_\infty(q^3)^{27}_\infty + q^4(q)^{48}_\infty(q^3)^{21}_\infty + q^4(q)^{66}_\infty(q^3)^{15}_\infty + q^7(q)^3_\infty(q^3)^{60}_\infty$$
$$+ q^7(q)^{12}_\infty(q^3)^{57}_\infty + q^7(q)^{21}_\infty(q^3)^{54}_\infty + q^8(q)^{54}_\infty(q^3)^{51}_\infty + q^8(q)^{198}_\infty(q^3)^3_\infty + q^9(q)^{15}_\infty(q^3)^{72}_\infty$$
$$+ q^9(q)^{51}_\infty(q^3)^{60}_\infty + q^9(q)^{195}_\infty(q^3)^{12}_\infty + q^{10}(q)^{255}_\infty$$

which, omitting the calculations,

$$\equiv q(q^3)^{13}_\infty + q^2(q^3)^{12}_\infty(q^9)^3_\infty + q^3(q^3)^{29}_\infty + q^4(q^3)^{10}_\infty(q^9)^9_\infty + q^4(q^3)^{19}_\infty(q^9)^6_\infty$$
$$+ q^4(q^3)^{28}_\infty(q^9)^3_\infty + q^5(q^3)^9_\infty(q^9)^{12}_\infty + q^6(q^3)^{26}_\infty(q^9)^9_\infty + q^7(q^3)^7_\infty(q^9)^{18}_\infty + q^7(q^3)^{16}_\infty(q^9)^{15}_\infty$$
$$+ q^7(q^3)^{61}_\infty + q^8(q^3)^6_\infty(q^9)^{21}_\infty + q^8(q^3)^{24}_\infty(q^9)^{15}_\infty + q^9(q^3)^{23}_\infty(q^9)^{18}_\infty + q^9(q^3)^{59}_\infty(q^9)^6_\infty$$
$$+ q^9(q^3)^{77}_\infty + q^{10}(q^3)^4_\infty(q^9)^{27}_\infty + q^{10}(q^3)^{22}_\infty(q^9)^{21}_\infty + q^{10}(q^3)^{31}_\infty(q^9)^{18}_\infty + q^{10}(q^3)^{58}_\infty(q^9)^9_\infty$$
$$+ q^{10}(q^3)^{76}_\infty(q^9)^3_\infty + q^{10}(q^3)^{85}_\infty + q^{11}(q^3)^3_\infty(q^9)^{30}_\infty + q^{11}(q^3)^{12}_\infty(q^9)^{27}_\infty + q^{11}(q^3)^{84}_\infty(q^9)^3_\infty$$
$$+ q^{12}(q^3)^{29}_\infty(q^9)^{24}_\infty + q^{12}(q^3)^{56}_\infty(q^9)^{15}_\infty + q^{13}(q^3)_\infty(q^9)^{36}_\infty + q^{13}(q^3)^{55}_\infty(q^9)^{18}_\infty + q^{13}(q^3)^{73}_\infty(q^9)^{12}_\infty$$
$$+ q^{14}(q^9)^{39}_\infty + q^{14}(q^3)^9_\infty(q^9)^{36}_\infty + q^{14}(q^3)^{27}_\infty(q^9)^{30}_\infty + q^{14}(q^3)^{54}_\infty(q^9)^{21}_\infty + q^{14}(q^3)^{72}_\infty(q^9)^{15}_\infty$$
$$+ q^{14}(q^3)^{81}_\infty(q^9)^{12}_\infty + q^{15}(q^3)^{80}_\infty(q^9)^{15}_\infty + q^{19}(q^3)^4_\infty(q^9)^{51}_\infty + q^{20}(q^3)^3_\infty(q^9)^{54}_\infty + q^{20}(q^3)^{12}_\infty(q^9)^{51}_\infty$$
$$+ q^{21}(q^3)^{11}_\infty(q^9)^{54}_\infty + q^{22}(q^3)_\infty(q^9)^{60}_\infty + q^{22}(q^3)^{10}_\infty(q^9)^{57}_\infty + q^{22}(q^3)^{19}_\infty(q^9)^{54}_\infty + q^{23}(q^9)^{63}_\infty$$

$$+ q^{24}(q^3)_\infty^8(q^9)_\infty^{63} + q^{24}(q^3)_\infty^{17}(q^9)_\infty^{60} + q^{24}(q^3)_\infty^{53}(q^9)_\infty^{48} + q^{25}(q^3)_\infty^{7}(q^9)_\infty^{66} + q^{25}(q^3)_\infty^{61}(q^9)_\infty^{48}$$
$$+ q^{26}(q^3)_\infty^6(q^9)_\infty^{69} + q^{26}(q^3)_\infty^{15}(q^9)_\infty^{66} + q^{26}(q^3)_\infty^{51}(q^9)_\infty^{54} + q^{26}(q^3)_\infty^{69}(q^9)_\infty^{48} + q^{26}(q^3)_\infty^{60}(q^9)_\infty^{51}$$
$$+ q^{27}(q^3)_\infty^{68}(q^9)_\infty^{51} + q^{30}(q^3)_\infty^{65}(q^9)_\infty^{60} + q^{31}(q^3)_\infty^{64}(q^9)_\infty^{63} + q^{72}(q^3)_\infty^{5}(q^9)_\infty^{192} + q^{73}(q^3)_\infty^{13}(q^9)_\infty^{192}$$
$$+ q^{74}(q^3)_\infty^3(q^9)_\infty^{198} + q^{74}(q^3)_\infty^{12}(q^9)_\infty^{195} + q^{74}(q^3)_\infty^{21}(q^9)_\infty^{192} + q^{75}(q^3)_\infty^{20}(q^9)_\infty^{195} + q^{78}(q^3)_\infty^{17}(q^9)_\infty^{204}$$
$$+ q^{79}(q^3)_\infty^{16}(q^9)_\infty^{207} + q^{90}(q^3)_\infty^{5}(q^9)_\infty^{240} + q^{91}(q^3)_\infty^{4}(q^9)_\infty^{243} + q^{94}(q^3)_\infty(q^9)_\infty^{252} + q^{95}(q^9)_\infty^{255}.$$

If we extract those terms in which the power is congruent to 2 (mod 3), divide by q^2 and replace q^3 by q, we find

$$\sum_{n\geq 0} a_{16}(243n+202)q^n$$
$$\equiv (q)_\infty^{12}(q^3)_\infty^3 + q(q)_\infty^9(q^3)_\infty^{12} + q^2(q)_\infty^6(q^3)_\infty^{21} + q^2(q)_\infty^{24}(q^3)_\infty^{15} + q^3(q)_\infty^3(q^3)_\infty^{30} + q^3(q)_\infty^{12}(q^3)_\infty^{27}$$
$$+ q^3(q)_\infty^{84}(q^3)_\infty^3 + q^4(q^3)_\infty^{39} + q^4(q)_\infty^9(q^3)_\infty^{36} + q^4(q)_\infty^{27}(q^3)_\infty^{30} + q^4(q)_\infty^{54}(q^3)_\infty^{21} + q^4(q)_\infty^{72}(q^3)_\infty^{15}$$
$$+ q^4(q)_\infty^{81}(q^3)_\infty^{12} + q^6(q)_\infty^3(q^3)_\infty^{54} + q^6(q)_\infty^{12}(q^3)_\infty^{51} + q^7(q^3)_\infty^{63} + q^8(q)_\infty^6(q^3)_\infty^{69} + q^8(q)_\infty^{15}(q^3)_\infty^{66}$$
$$+ q^8(q)_\infty^{51}(q^3)_\infty^{54} + q^8(q)_\infty^{60}(q^3)_\infty^{51} + q^8(q)_\infty^{69}(q^3)_\infty^{48} + q^{24}(q)_\infty^3(q^3)_\infty^{198} + q^{24}(q)_\infty^{12}(q^3)_\infty^{195}$$
$$+ q^{24}(q)_\infty^{21}(q^3)_\infty^{192} + q^{31}(q^3)_\infty^{255}$$

$$\equiv (q^3)_\infty^7 + q(q^3)_\infty^{15} + q^2(q^3)_\infty^{14}(q^9)_\infty^3 + q^3(q^3)_\infty^{13}(q^9)_\infty^6 + q^3(q^3)_\infty^{31} + q^4(q^3)_\infty^3(q^9)_\infty^{12}$$
$$+ q^4(q^3)_\infty^{12}(q^9)_\infty^9 + q^4(q^3)_\infty^{21}(q^9)_\infty^6 + q^4(q^3)_\infty^{30}(q^9)_\infty^3 + q^5(q^3)_\infty^{38}(q^9)_\infty^3 + q^6(q^3)_\infty^{37}(q^9)_\infty^6$$
$$+ q^7(q^3)_\infty^{54}(q^9)_\infty^3 + q^7(q^3)_\infty^{63} + q^8(q^3)_\infty^{71} + q^9(q^3)_\infty^{70}(q^9)_\infty^3 + q^{10}(q^3)_\infty^{15}(q^9)_\infty^{24} + q^{10}(q^3)_\infty^{51}(q^9)_\infty^{12}$$
$$+ q^{11}(q^3)_\infty^{23}(q^9)_\infty^{24} + q^{11}(q^3)_\infty^{68}(q^9)_\infty^9 + q^{12}(q^3)_\infty^{31}(q^9)_\infty^{24} + q^{12}(q^3)_\infty^{67}(q^9)_\infty^{12} + q^{14}(q^3)_\infty^{29}(q^9)_\infty^{30}$$
$$+ q^{14}(q^3)_\infty^{65}(q^9)_\infty^{18} + q^{15}(q^3)_\infty^{19}(q^9)_\infty^{36} + q^{15}(q^3)_\infty^{28}(q^9)_\infty^{33} + q^{15}(q^3)_\infty^{64}(q^9)_\infty^{21} + q^{19}(q^3)_\infty^{15}(q^9)_\infty^{48}$$
$$+ q^{20}(q^3)_\infty^{23}(q^9)_\infty^{48} + q^{21}(q^3)_\infty^{22}(q^9)_\infty^{51} + q^{23}(q^3)_\infty^{11}(q^9)_\infty^{60} + q^{23}(q^3)_\infty^{20}(q^9)_\infty^{57} + q^{24}(q^3)_\infty^{199}$$
$$+ q^{24}(q^3)_\infty^{55}(q^9)_\infty^{48} + q^{26}(q^3)_\infty^{53}(q^9)_\infty^{54} + q^{26}(q^3)_\infty^{197}(q^9)_\infty^{6} + q^{27}(q^3)_\infty^{7}(q^9)_\infty^{72} + q^{27}(q^3)_\infty^{52}(q^9)_\infty^{57}$$
$$+ q^{27}(q^3)_\infty^{196}(q^9)_\infty^{9} + q^{29}(q^3)_\infty^{14}(q^9)_\infty^{75} + q^{29}(q^3)_\infty^{50}(q^9)_\infty^{63} + q^{29}(q^3)_\infty^{194}(q^9)_\infty^{15} + q^{30}(q^3)_\infty^{13}(q^9)_\infty^{78}$$
$$+ q^{30}(q^3)_\infty^{49}(q^9)_\infty^{66} + q^{30}(q^3)_\infty^{193}(q^9)_\infty^{18} + q^{31}(q^3)_\infty^{3}(q^9)_\infty^{84} + q^{31}(q^3)_\infty^{12}(q^9)_\infty^{81} + q^{31}(q^3)_\infty^{48}(q^9)_\infty^{69}$$
$$+ q^{31}(q^3)_\infty^{192}(q^9)_\infty^{21} + q^{31}(q^3)_\infty^{255}.$$

If we extract those terms in which the power is $1 \pmod 3$, divide by q and replace q^3 by q, we obtain

$$\begin{aligned}
&\sum_{n\geq 0} a_{16}(729n+445)q^n \\
&\equiv (q)_\infty^{15} + q(q)_\infty^{3}(q^3)_\infty^{12} + q(q)_\infty^{12}(q^3)_\infty^{9} + q(q)_\infty^{21}(q^3)_\infty^{6} + q(q)_\infty^{30}(q^3)_\infty^{3} + q^2(q)_\infty^{54}(q^3)_\infty^{3} \\
&\ + q^2(q)_\infty^{63} + q^3(q)_\infty^{15}(q^3)_\infty^{24} + q^3(q)_\infty^{51}(q^3)_\infty^{12} + q^6(q)_\infty^{15}(q^3)_\infty^{48} + q^{10}(q)_\infty^{3}(q^3)_\infty^{84} + q^{10}(q)_\infty^{12}(q^3)_\infty^{81} \\
&\ + q^{10}(q)_\infty^{48}(q^3)_\infty^{69} + q^{10}(q)_\infty^{192}(q^3)_\infty^{21} + q^{10}(q)_\infty^{255} \\
&\equiv (q^3)_\infty^{5} + q(q^3)_\infty^{4}(q^9)_\infty^{3} + q^3(q^3)_\infty^{20}(q^9)_\infty^{3} + q^4(q^3)_\infty(q^9)_\infty^{12} + q^4(q^3)_\infty^{10}(q^9)_\infty^{9} + q^4(q^3)_\infty^{19}(q^9)_\infty^{6} \\
&\ + q^5(q^9)_\infty^{15} + q^6(q^3)_\infty^{8}(q^9)_\infty^{15} + q^6(q^3)_\infty^{17}(q^9)_\infty^{12} + q^6(q^3)_\infty^{53} + q^7(q^3)_\infty^{7}(q^9)_\infty^{18} + q^7(q^3)_\infty^{16}(q^9)_\infty^{15} \\
&\ + q^7(q^3)_\infty^{25}(q^9)_\infty^{12} + q^7(q^3)_\infty^{52}(q^9)_\infty^{3} + q^8(q^3)_\infty^{6}(q^9)_\infty^{21} + q^8(q^3)_\infty^{24}(q^9)_\infty^{15} + q^9(q^3)_\infty^{5}(q^9)_\infty^{24} \\
&\ + q^{10}(q^3)_\infty^{49}(q^9)_\infty^{12} + q^{10}(q^3)_\infty^{85} + q^{11}(q^3)_\infty^{3}(q^9)_\infty^{30} + q^{11}(q^3)_\infty^{48}(q^9)_\infty^{15} + q^{15}(q^3)_\infty^{80}(q^9)_\infty^{15} \\
&\ + q^{19}(q^3)_\infty^{4}(q^9)_\infty^{51} + q^{19}(q^3)_\infty^{13}(q^9)_\infty^{48} + q^{20}(q^3)_\infty^{3}(q^9)_\infty^{54} + q^{20}(q^3)_\infty^{12}(q^9)_\infty^{51} + q^{22}(q^3)_\infty(q^9)_\infty^{60} \\
&\ + q^{23}(q^9)_\infty^{63} + q^{27}(q^3)_\infty^{68}(q^9)_\infty^{51} + q^{30}(q^3)_\infty^{65}(q^9)_\infty^{60} + q^{31}(q^3)_\infty^{64}(q^9)_\infty^{63} + q^{75}(q^3)_\infty^{20}(q^9)_\infty^{195} \\
&\ + q^{78}(q^3)_\infty^{17}(q^9)_\infty^{204} + q^{79}(q^3)_\infty^{16}(q^9)_\infty^{207} + q^{90}(q^3)_\infty^{5}(q^9)_\infty^{240} + q^{91}(q^3)_\infty^{4}(q^9)_\infty^{243} + q^{94}(q^3)_\infty(q^9)_\infty^{252} \\
&\ + q^{95}(q^9)_\infty^{255}.
\end{aligned}$$

If we extract those terms in which the power is $2 \pmod 3$, divide by q^2 and replace q^3 by q, we obtain

$$\begin{aligned}
&\sum_{n\geq 0} a_{16}(2187n+1903)q^n \\
&\equiv q(q^3)_\infty^{15} + q^2(q)_\infty^{6}(q^3)_\infty^{21} + q^2(q)_\infty^{24}(q^3)_\infty^{15} + q^3(q)_\infty^{3}(q^3)_\infty^{30} + q^3(q)_\infty^{48}(q^3)_\infty^{15} + q^6(q)_\infty^{3}(q^3)_\infty^{54} \\
&\ + q^6(q)_\infty^{12}(q^3)_\infty^{51} + q^7(q^3)_\infty^{63} + q^{31}(q^3)_\infty^{255} \\
&\equiv q(q^3)_\infty^{15} + q^4(q^3)_\infty^{21}(q^9)_\infty^{6} + q^4(q^3)_\infty^{30}(q^9)_\infty^{3} + q^7(q^3)_\infty^{54}(q^9)_\infty^{3} + q^7(q^3)_\infty^{63} + q^{10}(q^3)_\infty^{15}(q^9)_\infty^{24} \\
&\ + q^{10}(q^3)_\infty^{51}(q^9)_\infty^{12} + q^{19}(q^3)_\infty^{15}(q^9)_\infty^{48} + q^{31}(q^3)_\infty^{255},
\end{aligned}$$

Since the only powers appearing are $1 \pmod 3$, we have

$$\sum_{n\geq 0} a_{16}(6561n+1903)q^n \equiv 0 \text{ and } \sum_{n\geq 0} a_{16}(6561n+6277)q^n \equiv 0.$$

Also

$$
\begin{aligned}
&\sum_{n\geq 0} a_{16}(6561n+4090)q^n\\
&\equiv (q)_\infty^{15} + q(q)_\infty^{21}(q^3)_\infty^{6} + q(q)_\infty^{30}(q^3)_\infty^{3} + q^2(q)_\infty^{54}(q^3)_\infty^{3} + q^2(q)_\infty^{63} + q^3(q)_\infty^{15}(q^3)_\infty^{24}\\
&\quad + q^3(q)_\infty^{51}(q^3)_\infty^{12} + q^6(q)_\infty^{15}(q^3)_\infty^{48} + q^{10}(q)_\infty^{255}
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)_\infty^{5} + q(q^3)_\infty^{4}(q^9)_\infty^{3} + q^2(q^3)_\infty^{12}(q^9)_\infty^{3} + q^3(q^3)_\infty^{20}(q^9)_\infty^{3} + q^4(q^3)_\infty(q^9)_\infty^{12} + q^4(q^3)_\infty^{10}(q^9)_\infty^{9}\\
&\quad + q^4(q^3)_\infty^{19}(q^9)_\infty^{6} + q^5(q^9)_\infty^{15} + q^5(q^3)_\infty^{9}(q^9)_\infty^{12} + q^6(q^3)_\infty^{8}(q^9)_\infty^{15} + q^6(q^3)_\infty^{17}(q^9)_\infty^{12} + q^6(q^3)_\infty^{53}\\
&\quad + q^7(q^3)_\infty^{7}(q^9)_\infty^{18} + q^7(q^3)_\infty^{16}(q^9)_\infty^{15} + q^7(q^3)_\infty^{25}(q^9)_\infty^{12} + q^7(q^3)_\infty^{52}(q^9)_\infty^{3} + q^8(q^3)_\infty^{6}(q^9)_\infty^{21}\\
&\quad + q^8(q^3)_\infty^{24}(q^9)_\infty^{15} + q^9(q^3)_\infty^{5}(q^9)_\infty^{24} + q^{10}(q^3)_\infty^{49}(q^9)_\infty^{12} + q^{10}(q^3)_\infty^{85} + q^{11}(q^3)_\infty^{3}(q^9)_\infty^{30}\\
&\quad + q^{11}(q^3)_\infty^{48}(q^9)_\infty^{15} + q^{11}(q^3)_\infty^{84}(q^9)_\infty^{3} + q^{14}(q^3)_\infty^{81}(q^9)_\infty^{12} + q^{15}(q^3)_\infty^{80}(q^9)_\infty^{15} + q^{19}(q^3)_\infty^{4}(q^9)_\infty^{51}\\
&\quad + q^{19}(q^3)_\infty^{13}(q^9)_\infty^{48} + q^{20}(q^3)_\infty^{3}(q^9)_\infty^{54} + q^{20}(q^3)_\infty^{12}(q^9)_\infty^{51} + q^{22}(q^3)_\infty(q^9)_\infty^{60} + q^{23}(q^9)_\infty^{63}\\
&\quad + q^{26}(q^3)_\infty^{69}(q^9)_\infty^{48} + q^{27}(q^3)_\infty^{68}(q^9)_\infty^{51} + q^{30}(q^3)_\infty^{65}(q^9)_\infty^{60} + q^{31}(q^3)_\infty^{64}(q^9)_\infty^{63} + q^{74}(q^3)_\infty^{21}(q^9)_\infty^{192}\\
&\quad + q^{75}(q^3)_\infty^{20}(q^9)_\infty^{195} + q^{78}(q^3)_\infty^{17}(q^9)_\infty^{204} + q^{79}(q^3)_\infty^{16}(q^9)_\infty^{207} + q^{90}(q^3)_\infty^{5}(q^9)_\infty^{240} + q^{91}(q^3)_\infty^{4}(q^9)_\infty^{243}\\
&\quad + q^{94}(q^3)_\infty(q^9)_\infty^{252} + q^{95}(q^9)_\infty^{255}.
\end{aligned}
$$

If we now extract those terms in which the power is 2 (mod 3), divide by q^2 and replace q^3 by q, we find

$$
\begin{aligned}
&\sum_{n\geq 0} a_{16}(19683n+17212)q^n\\
&\equiv (q)_\infty^{12}(q^3)_\infty^{3} + q(q^3)_\infty^{15} + q(q)_\infty^{9}(q^3)_\infty^{12} + q^2(q)_\infty^{6}(q^3)_\infty^{21} + q^2(q)_\infty^{24}(q^3)_\infty^{15} + q^3(q)_\infty^{3}(q^3)_\infty^{30}\\
&\quad + q^3(q)_\infty^{48}(q^3)_\infty^{15} + q^3(q)_\infty^{84}(q^3)_\infty^{3} + q^4(q)_\infty^{81}(q^3)_\infty^{12} + q^6(q)_\infty^{3}(q^3)_\infty^{54} + q^6(q)_\infty^{12}(q^3)_\infty^{51} + q^7(q^3)_\infty^{63}\\
&\quad + q^8(q)_\infty^{69}(q^3)_\infty^{48} + q^{24}(q)_\infty^{21}(q^3)_\infty^{192} + q^{31}(q^3)_\infty^{255}
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (q^3)_\infty^{7} + q^2(q^3)_\infty^{14}(q^9)_\infty^{3} + q^3(q^3)_\infty^{13}(q^9)_\infty^{6} + q^3(q^3)_\infty^{31} + q^4(q^3)_\infty^{3}(q^9)_\infty^{12} + q^4(q^3)_\infty^{12}(q^9)_\infty^{9}\\
&\quad + q^4(q^3)_\infty^{21}(q^9)_\infty^{6} + q^4(q^3)_\infty^{30}(q^9)_\infty^{3} + q^4(q^3)_\infty^{39} + q^5(q^3)_\infty^{38}(q^9)_\infty^{3} + q^6(q^3)_\infty^{37}(q^9)_\infty^{6}\\
&\quad + q^7(q^3)_\infty^{27}(q^9)_\infty^{12} + q^7(q^3)_\infty^{36}(q^9)_\infty^{9} + q^7(q^3)_\infty^{54}(q^9)_\infty^{3} + q^7(q^3)_\infty^{63} + q^8(q^3)_\infty^{71} + q^9(q^3)_\infty^{70}(q^9)_\infty^{3}\\
&\quad + q^{10}(q^3)_\infty^{15}(q^9)_\infty^{24} + q^{10}(q^3)_\infty^{51}(q^9)_\infty^{12} + q^{10}(q^3)_\infty^{69}(q^9)_\infty^{6} + q^{11}(q^3)^{23}(q^9)_\infty^{24} + q^{11}(q^3)_\infty^{68}(q^9)_\infty^{9}
\end{aligned}
$$

$$+ q^{12}(q^3)_\infty^{31}(q^9)_\infty^{24} + q^{12}(q^3)_\infty^{67}(q^9)_\infty^{12} + q^{13}(q^3)_\infty^{30}(q^9)_\infty^{27} + q^{13}(q^3)_\infty^{66}(q^9)_\infty^{15} + q^{14}(q^3)_\infty^{29}(q^9)_\infty^{30}$$
$$+ q^{14}(q^3)_\infty^{65}(q^9)_\infty^{18} + q^{15}(q^3)_\infty^{19}(q^9)_\infty^{36} + q^{15}(q^3)_\infty^{28}(q^9)_\infty^{33} + q^{15}(q^3)_\infty^{64}(q^9)_\infty^{21} + q^{20}(q^3)_\infty^{23}(q^9)_\infty^{48}$$
$$+ q^{21}(q^3)_\infty^{22}(q^9)_\infty^{51} + q^{22}(q^3)_\infty^{21}(q^9)_\infty^{54} + q^{23}(q^3)_\infty^{11}(q^9)_\infty^{60} + q^{23}(q^3)_\infty^{20}(q^9)_\infty^{57} + q^{24}(q^3)_\infty^{55}(q^9)_\infty^{48}$$
$$+ q^{24}(q^3)_\infty^{199} + q^{25}(q^3)_\infty^{54}(q^9)_\infty^{51} + q^{25}(q^3)_\infty^{198}(q^9)_\infty^{3} + q^{26}(q^3)_\infty^{53}(q^9)_\infty^{54} + q^{26}(q^3)_\infty^{197}(q^9)_\infty^{6}$$
$$+ q^{27}(q^3)_\infty^{7}(q^9)_\infty^{72} + q^{27}(q^3)_\infty^{52}(q^9)_\infty^{57} + q^{27}(q^3)_\infty^{196}(q^9)_\infty^{9} + q^{28}(q^3)_\infty^{15}(q^9)_\infty^{72} + q^{28}(q^3)_\infty^{51}(q^9)_\infty^{60}$$
$$+ q^{28}(q^3)_\infty^{195}(q^9)_\infty^{12} + q^{29}(q^3)_\infty^{14}(q^9)_\infty^{75} + q^{29}(q^3)_\infty^{50}(q^9)_\infty^{63} + q^{29}(q^3)_\infty^{194}(q^9)_\infty^{15} + q^{30}(q^3)_\infty^{13}(q^9)_\infty^{78}$$
$$+ q^{30}(q^3)_\infty^{49}(q^9)_\infty^{66} + q^{30}(q^3)_\infty^{193}(q^9)_\infty^{18} + q^{31}(q^3)_\infty^{3}(q^9)_\infty^{84} + q^{31}(q^3)_\infty^{12}(q^9)_\infty^{81} + q^{31}(q^3)_\infty^{48}(q^9)_\infty^{69}$$
$$+ q^{31}(q^3)_\infty^{192}(q^9)_\infty^{21} + q^{31}(q^3)_\infty^{255}.$$

If we extract those terms in which the power is 1 (mod 3), divide by q and replace q^3 by q, we obtain

$$\sum_{n\geq 0} a_{16}(59049n + 36895)q^n$$
$$\equiv q(q)_\infty^{3}(q^3)_\infty^{12} + q(q)_\infty^{12}(q^3)_\infty^{9} + q(q)_\infty^{21}(q^3)_\infty^{6} + q(q)_\infty^{30}(q^3)_\infty^{3} + q(q)_\infty^{39} + q^2(q)_\infty^{27}(q^3)_\infty^{12}$$
$$+ q^2(q)_\infty^{36}(q^3)_\infty^{9} + q^2(q)_\infty^{54}(q^3)_\infty^{3} + q^2(q)_\infty^{63} + q^3(q)_\infty^{15}(q^3)_\infty^{24} + q^3(q)_\infty^{51}(q^3)_\infty^{12} + q^3(q)_\infty^{69}(q^3)_\infty^{6}$$
$$+ q^4(q)_\infty^{30}(q^3)_\infty^{27} + q^4(q)_\infty^{66}(q^3)_\infty^{15} + q^7(q)_\infty^{21}(q^3)_\infty^{54} + q^8(q)_\infty^{54}(q^3)_\infty^{51} + q^8(q)_\infty^{198}(q^3)_\infty^{3} + q^9(q)_\infty^{15}(q^3)_\infty^{72}$$
$$+ q^9(q)_\infty^{51}(q^3)_\infty^{60} + q^9(q)_\infty^{195}(q^3)_\infty^{12} + q^{10}(q)_\infty^{3}(q^3)_\infty^{84} + q^{10}(q)_\infty^{12}(q^3)_\infty^{81} + q^{10}(q)_\infty^{48}(q^3)_\infty^{69}$$
$$+ q^{10}(q)_\infty^{192}(q^3)_\infty^{21} + q^{10}(q)_\infty^{255}$$

$$\equiv q(q^3)_\infty^{13} + q^2(q^3)_\infty^{12}(q^9)_\infty^{3} + q^3(q^3)_\infty^{29} + q^4(q^3)_\infty^{10}(q^9)_\infty^{9} + q^4(q^3)_\infty^{19}(q^9)_\infty^{6} + q^4(q^3)_\infty^{28}(q^9)_\infty^{3}$$
$$+ q^5(q^3)_\infty^{9}(q^8)_\infty^{12} + q^5(q^3)_\infty^{27}(q^9)_\infty^{6} + q^6(q^3)_\infty^{26}(q^9)_\infty^{9} + q^7(q^3)_\infty^{7}(q^9)_\infty^{18} + q^7(q^3)_\infty^{16}(q^9)_\infty^{15}$$
$$+ q^7(q^3)_\infty^{61} + q^8(q^3)_\infty^{6}(q^9)_\infty^{21} + q^8(q^3)_\infty^{33}(q^9)_\infty^{12} + q^8(q^3)_\infty^{60}(q^9)_\infty^{3} + q^9(q^3)_\infty^{23}(q^9)_\infty^{18}$$
$$+ q^9(q^3)_\infty^{59}(q^9)_\infty^{6} + q^9(q^3)_\infty^{77} + q^{10}(q^3)_\infty^{4}(q^9)_\infty^{27} + q^{10}(q^3)_\infty^{22}(q^9)_\infty^{21} + q^{10}(q^3)_\infty^{31}(q^9)_\infty^{18}$$
$$+ q^{10}(q^3)_\infty^{58}(q^9)_\infty^{9} + q^{10}(q^3)_\infty^{76}(q^9)_\infty^{3} + q^{10}(q^3)_\infty^{85} + q^{11}(q^3)_\infty^{3}(q^9)_\infty^{30} + q^{11}(q^3)_\infty^{12}(q^9)_\infty^{27}$$
$$+ q^{11}(q^3)_\infty^{57}(q^9)_\infty^{12} + q^{12}(q^3)_\infty^{29}(q^9)_\infty^{24} + q^{12}(q^3)_\infty^{56}(q^9)_\infty^{15} + q^{13}(q^3)_\infty(q^9)_\infty^{36} + q^{13}(q^3)_\infty^{55}(q^9)_\infty^{18}$$
$$+ q^{13}(q^3)_\infty^{73}(q^9)_\infty^{12} + q^{14}(q^9)_\infty^{39} + q^{14}(q^3)_\infty^{9}(q^9)_\infty^{36} + q^{14}(q^3)_\infty^{27}(q^9)_\infty^{30} + q^{14}(q^3)_\infty^{54}(q^9)_\infty^{21}$$
$$+ q^{14}(q^3)_\infty^{72}(q^9)_\infty^{15} + q^{15}(q^3)_\infty^{80}(q^9)_\infty^{15} + q^{19}(q^3)_\infty^{4}(q^9)_\infty^{51} + q^{20}(q^3)_\infty^{3}(q^9)_\infty^{54} + q^{20}(q^3)_\infty^{21}(q^9)_\infty^{48}$$
$$+ q^{21}(q^3)_\infty^{11}(q^9)_\infty^{54} + q^{22}(q^3)_\infty(q^9)_\infty^{60} + q^{22}(q^3)_\infty^{10}(q^9)_\infty^{57} + q^{22}(q^3)_\infty^{19}(q^9)_\infty^{54} + q^{23}(q^9)_\infty^{63}$$

$$+ q^{23}(q^3)^9_\infty(q^9)^{60}_\infty + q^{24}(q^3)^8_\infty(q^9)^{63}_\infty + q^{24}(q^3)^{17}_\infty(q^9)^{60}_\infty + q^{24}(q^3)^{53}_\infty(q^9)^{48}_\infty + q^{25}(q^3)^7_\infty(q^9)^{66}_\infty$$
$$+ q^{25}(q^3)^{61}_\infty(q^9)^{48}_\infty + q^{26}(q^3)^6_\infty(q^9)^{69}_\infty + q^{26}(q^3)^{15}_\infty(q^9)^{66}_\infty + q^{26}(q^3)^{51}_\infty(q^9)^{54}_\infty + q^{26}(q^3)^{60}_\infty(q^9)^{51}_\infty$$
$$+ q^{27}(q^3)^{68}_\infty(q^9)^{51}_\infty + q^{30}(q^3)^{65}_\infty(q^9)^{60}_\infty + q^{31}(q^3)^{64}_\infty(q^9)^{63}_\infty + q^{72}(q^3)^5_\infty(q^9)^{192}_\infty + q^{73}(q^3)^{13}_\infty(q^9)^{192}_\infty$$
$$+ q^{74}(q^3)^3_\infty(q^9)^{198}_\infty + q^{74}(q^3)^{12}_\infty(q^9)^{195}_\infty + q^{75}(q^3)^{20}_\infty(q^9)^{195}_\infty + q^{78}(q^3)^{17}_\infty(q^9)^{204}_\infty + q^{79}(q^3)^{16}_\infty(q^9)^{207}_\infty$$
$$+ q^{90}(q^3)^5_\infty(q^9)^{240}_\infty + q^{91}(q^3)^4_\infty(q^9)^{243}_\infty + q^{94}(q^3)_\infty(q^9)^{252}_\infty + q^{95}(q^9)^{255}_\infty.$$

If we extract those terms in which the power is 2 (mod 3), divide by q^2 and replace q^3 by q, we find

$$\sum_{n\geq 0} a_{16}(177147n + 154993)q^n$$
$$\equiv (q)^{12}_\infty(q^3)^3_\infty + q(q)^9_\infty(q^3)^{12}_\infty + q(q)^{27}_\infty(q^3)^6_\infty + q^2(q)^6_\infty(q^3)^{21}_\infty + q^2(q)^{33}_\infty(q^3)^{12}_\infty + q^2(q)^{60}_\infty(q^3)^3_\infty$$
$$+ q^3(q)^3_\infty(q^3)^{30}_\infty + q^3(q)^{12}_\infty(q^3)^{27}_\infty + q^3(q)^{57}_\infty(q^3)^{12}_\infty + q^4(q^3)^{39}_\infty + q^4(q)^9_\infty(q^3)^{36}_\infty + q^4(q)^{27}_\infty(q^3)^{30}_\infty$$
$$+ q^4(q)^{54}_\infty(q^3)^{21}_\infty + q^4(q)^{72}_\infty(q^3)^{15}_\infty + q^6(q)^3_\infty(q^3)^{54}_\infty + q^6(q)^{21}_\infty(q^3)^{48}_\infty + q^7(q^3)^{63}_\infty + q^7(q)^9_\infty(q^3)^{60}_\infty$$
$$+ q^8(q)^6_\infty(q^3)^{69}_\infty + q^8(q)^{15}_\infty(q^3)^{66}_\infty + q^8(q)^{51}_\infty(q^3)^{54}_\infty + q^8(q)^{60}_\infty(q^3)^{51}_\infty + q^{24}(q)^3_\infty(q^3)^{198}_\infty$$
$$+ q^{24}(q)^{12}_\infty(q^3)^{195}_\infty + q^{31}(q^3)^{255}_\infty$$

$$\equiv (q^3)^7_\infty + q^2(q^3)^{23}_\infty + q^3(q^3)^{13}_\infty(q^9)^6_\infty + q^3(q^3)^{22}_\infty(q^9)^3_\infty + q^3(q^3)^{31}_\infty + q^4(q^3)^3_\infty(q^9)^{12}_\infty$$
$$+ q^4(q^3)^{12}_\infty(q^9)^9_\infty + q^4(q^3)^{39}_\infty + q^5(q^3)^{20}_\infty(q^9)^9_\infty + q^5(q^3)^{29}_\infty(q^9)^6_\infty + q^6(q^3)^{19}_\infty(q^9)^{12}_\infty$$
$$+ q^6(q^3)^{28}_\infty(q^9)^9_\infty + q^7(q^3)^{27}_\infty(q^9)^{12}_\infty + q^7(q^3)^{36}_\infty(q^9)^9_\infty + q^8(q^3)^{53}_\infty(q^9)^6_\infty + q^8(q^3)^{62}_\infty(q^9)^3_\infty$$
$$+ q^9(q^3)^7_\infty(q^9)^{24}_\infty + q^9(q^3)^{52}_\infty(q^9)^9_\infty + q^9(q^3)^{61}_\infty(q^9)^6_\infty + q^{10}(q^3)^6_\infty(q^9)^{27}_\infty + q^{10}(q^3)^{15}_\infty(q^9)^{24}_\infty$$
$$+ q^{10}(q^3)^{51}_\infty(q^9)^{12}_\infty + q^{10}(q^3)^{60}_\infty(q^9)^9_\infty + q^{10}(q^3)^{69}_\infty(q^9)^6_\infty + q^{11}(q^3)^{14}_\infty(q^9)^{27}_\infty + q^{11}(q^3)^{50}_\infty(q^9)^{15}_\infty$$
$$+ q^{12}(q^3)^{13}_\infty(q^9)^{30}_\infty + q^{12}(q^3)^{49}_\infty(q^9)^{18}_\infty + q^{13}(q^3)^{12}_\infty(q^9)^{33}_\infty + q^{13}(q^3)^{30}_\infty(q^9)^{27}_\infty + q^{13}(q^3)^{48}_\infty(q^9)^{21}_\infty$$
$$+ q^{13}(q^3)^{66}_\infty(q^9)^{15}_\infty + q^{18}(q^3)^7_\infty(q^9)^{48}_\infty + q^{19}(q^3)^{15}_\infty(q^9)^{48}_\infty + q^{20}(q^3)^{14}_\infty(q^9)^{51}_\infty + q^{21}(q^3)^{13}_\infty(q^9)^{54}_\infty$$
$$+ q^{22}(q^3)^3_\infty(q^9)^{60}_\infty + q^{22}(q^3)^{12}_\infty(q^9)^{57}_\infty + q^{22}(q^3)^{21}_\infty(q^9)^{54}_\infty + q^{25}(q^3)^{54}_\infty(q^9)^{51}_\infty + q^{25}(q^3)^{198}_\infty(q^9)^3_\infty$$
$$+ q^{28}(q^3)^{15}_\infty(q^9)^{72}_\infty + q^{28}(q^3)^{51}_\infty(q^9)^{60}_\infty + q^{28}(q^3)^{195}_\infty(q^9)^{12}_\infty + q^{31}(q^3)^{255}_\infty.$$

If we extract those terms in which the power is 1 (mod 3), divide by q and replace q^3 by q, we obtain

$$\sum_{n\geq 0} a_{16}(531441n + 332140)q^n$$

$$\begin{aligned}
&\equiv q(q)_\infty^3(q^3)_\infty^{12}+q(q)_\infty^{12}(q^3)_\infty^9+q(q)_\infty^{39}+q^2(q)_\infty^{27}(q^3)_\infty^{12}+q^2(q)_\infty^{36}(q^3)_\infty^9+q^3(q)_\infty^6(q^3)_\infty^{27}\\
&\quad+q^3(q)_\infty^{15}(q^3)_\infty^{24}+q^3(q)_\infty^{51}(q^3)_\infty^{12}+q^3(q)_\infty^{60}(q^3)_\infty^9+q^3(q)_\infty^{69}(q^3)_\infty^6+q^4(q)_\infty^{12}(q^3)_\infty^{33}+q^4(q)_\infty^{30}(q^3)_\infty^{27}\\
&\quad+q^4(q)_\infty^{48}(q^3)_\infty^{21}+q^4(q)_\infty^{66}(q^3)_\infty^{15}+q^6(q)_\infty^{15}(q^3)_\infty^{48}+q^7(q)_\infty^3(q^3)_\infty^{60}+q^7(q)_\infty^{12}(q^3)_\infty^{57}+q^7(q)_\infty^{21}(q^3)_\infty^{54}\\
&\quad+q^8(q)_\infty^{54}(q^3)_\infty^{51}+q^8(q)_\infty^{198}(q^3)_\infty^3+q^9(q)_\infty^{15}(q^3)_\infty^{72}+q^9(q)_\infty^{51}(q^3)_\infty^{60}+q^9(q)_\infty^{195}(q^3)_\infty^{12}+q^{10}(q)_\infty^{255}
\end{aligned}$$

$$\begin{aligned}
&\equiv q(q^3)_\infty^{13}+q^3(q^3)_\infty^{20}(q^9)_\infty^3+q^3(q^3)_\infty^{29}+q^4(q^3)_\infty^{28}(q^9)_\infty^3+q^6(q^3)_\infty^8(q^9)_\infty^{15}+q^6(q^3)_\infty^{17}(q^9)_\infty^{12}\\
&\quad+q^6(q^3)_\infty^{26}(q^9)_\infty^9+q^6(q^3)_\infty^{53}+q^7(q^3)_\infty^{25}(q^9)_\infty^{12}+q^7(q^3)_\infty^{52}(q^9)_\infty^3+q^7(q^3)_\infty^{61}+q^9(q^3)_\infty^5(q^9)_\infty^{24}\\
&\quad+q^9(q^3)_\infty^{23}(q^9)_\infty^{18}+q^9(q^3)_\infty^{59}(q^9)_\infty^6+q^9(q^3)_\infty^{77}+q^{10}(q^3)_\infty^4(q^9)_\infty^{27}+q^{10}(q^3)_\infty^{22}(q^9)_\infty^{21}\\
&\quad+q^{10}(q^3)_\infty^{31}(q^9)_\infty^{18}+q^{10}(q^3)_\infty^{49}(q^9)_\infty^{12}+q^{10}(q^3)_\infty^{58}(q^9)_\infty^9+q^{10}(q^3)_\infty^{76}(q^9)_\infty^3+q^{10}(q^9)_\infty^{85}\\
&\quad+q^{11}(q^3)_\infty^{12}(q^9)_\infty^{27}+q^{11}(q^3)_\infty^{48}(q^9)_\infty^{15}+q^{11}(q^3)_\infty^{84}(q^9)_\infty^3+q^{12}(q^3)_\infty^{29}(q^9)_\infty^{24}+q^{12}(q^3)_\infty^{56}(q^9)_\infty^{15}\\
&\quad+q^{13}(q^3)_\infty(q^9)_\infty^{36}+q^{13}(q^3)_\infty^{55}(q^9)_\infty^{18}+q^{13}(q^3)_\infty^{73}(q^9)_\infty^{12}+q^{14}(q^9)_\infty^{39}+q^{14}(q^3)_\infty^9(q^9)_\infty^{36}\\
&\quad+q^{14}(q^3)_\infty^{27}(q^9)_\infty^{30}+q^{14}(q^3)_\infty^{54}(q^9)_\infty^{21}+q^{14}(q^3)_\infty^{72}(q^9)_\infty^{15}+q^{14}(q^3)_\infty^{81}(q^9)_\infty^{12}+q^{15}(q^3)_\infty^{80}(q^9)_\infty^{15}\\
&\quad+q^{19}(q^3)_\infty^{13}(q^9)_\infty^{48}+q^{21}(q^3)_\infty^{11}(q^9)_\infty^{54}+q^{22}(q^3)_\infty^{10}(q^9)_\infty^{57}+q^{22}(q^3)_\infty^{19}(q^9)_\infty^{54}+q^{24}(q^3)_\infty^8(q^9)_\infty^{63}\\
&\quad+q^{24}(q^3)_\infty^{17}(q^9)_\infty^{60}+q^{24}(q^3)_\infty^{53}(q^9)_\infty^{48}+q^{25}(q^3)_\infty^7(q^9)_\infty^{66}+q^{25}(q^3)_\infty^{61}(q^9)_\infty^{48}+q^{26}(q^3)_\infty^6(q^9)_\infty^{69}\\
&\quad+q^{26}(q^3)_\infty^{15}(q^9)_\infty^{66}+q^{26}(q^3)_\infty^{51}(q^9)_\infty^{54}+q^{26}(q^3)_\infty^{60}(q^9)_\infty^{51}+q^{26}(q^3)_\infty^{69}(q^9)_\infty^{48}+q^{27}(q^3)_\infty^{68}(q^9)_\infty^{51}\\
&\quad+q^{30}(q^3)_\infty^{65}(q^9)_\infty^{60}+q^{31}(q^3)_\infty^{64}(q^9)_\infty^{63}+q^{72}(q^3)_\infty^5(q^9)_\infty^{192}+q^{73}(q^3)_\infty^{13}(q^9)_\infty^{192}+q^{74}(q^3)_\infty^3(q^9)_\infty^{198}\\
&\quad+q^{74}(q^3)_\infty^{12}(q^9)_\infty^{195}+q^{74}(q^3)_\infty^{21}(q^9)_\infty^{192}+q^{75}(q^3)_\infty^{20}(q^9)_\infty^{195}+q^{78}(q^3)_\infty^{17}(q^9)_\infty^{204}+q^{79}(q^3)_\infty^{16}(q^9)_\infty^{207}\\
&\quad+q^{90}(q^3)_\infty^5(q^9)_\infty^{240}+q^{91}(q^3)_\infty^4(q^9)_\infty^{243}+q^{94}(q^3)_\infty(q^9)_\infty^{252}+q^{95}(q^9)_\infty^{255}.
\end{aligned}$$

If we extract those terms in which the power is 2 (mod 3), divide by q^2 and replace q^3 by q, we obtain

$$\begin{aligned}
&\sum_{n\geq 0} a_{16}(1594323n+1395022)q^n\\
&\equiv q^3(q)_\infty^{12}(q^3)_\infty^{27}+q^3(q)_\infty^{48}(q^3)_\infty^{15}+q^3(q)_\infty^{84}(q^3)_\infty^3+q^4(q^3)_\infty^{39}+q^4(q)_\infty^9(q^3)_\infty^{36}+q^4(q)_\infty^{27}(q^3)_\infty^{30}\\
&\quad+q^4(q)_\infty^{54}(q^3)_\infty^{21}+q^4(q)_\infty^{72}(q^3)_\infty^{15}+q^4(q)_\infty^{81}(q^3)_\infty^{12}+q^8(q)_\infty^6(q^3)_\infty^{69}+q^8(q)_\infty^{15}(q^3)_\infty^{66}+q^8(q)_\infty^{51}(q^3)_\infty^{54}\\
&\quad+q^8(q)_\infty^{60}(q^3)_\infty^{51}+q^8(q)_\infty^{69}(q^3)_\infty^{48}+q^{24}(q)_\infty^3(q^3)_\infty^{198}+q^{24}(q)_\infty^{12}(q^3)_\infty^{195}+q^{24}(q)_\infty^{21}(q^3)_\infty^{192}\\
&\quad+q^{31}(q^3)_\infty^{255}
\end{aligned}$$

$$\equiv q^3(q^3)_\infty^{31} + q^5(q^3)_\infty^{38}(q^9)_\infty^3 + q^6(q^3)_\infty^{37}(q^9)_\infty^6 + q^8(q^3)_\infty^{71} + q^9(q^3)_\infty^{70}(q^9)_\infty^3 + q^{11}(q^3)_\infty^{23}(q^9)_\infty^{24}$$
$$+ q^{11}(q^3)_\infty^{68}(q^9)_\infty^9 + q^{12}(q^3)_\infty^{31}(q^9)_\infty^{24} + q^{12}(q^3)_\infty^{67}(q^9)_\infty^{12} + q^{14}(q^3)_\infty^{29}(q^9)_\infty^{30} + q^{14}(q^3)_\infty^{65}(q^9)_\infty^{18}$$
$$+ q^{15}(q^3)_\infty^{19}(q^9)_\infty^{36} + q^{15}(q^3)_\infty^{28}(q^9)_\infty^{33} + q^{15}(q^3)_\infty^{64}(q^9)_\infty^{21} + q^{20}(q^3)_\infty^{23}(q^9)_\infty^{48} + q^{21}(q^3)_\infty^{22}(q^9)_\infty^{51}$$
$$+ q^{23}(q^3)_\infty^{11}(q^9)_\infty^{60} + q^{23}(q^3)_\infty^{20}(q^9)_\infty^{57} + q^{24}(q^3)_\infty^{55}(q^9)_\infty^{48} + q^{24}(q^3)_\infty^{199} + q^{26}(q^3)_\infty^{53}(q^9)_\infty^{54}$$
$$+ q^{26}(q^3)_\infty^{197}(q^9)_\infty^6 + q^{27}(q^3)_\infty^7(q^9)_\infty^{72} + q^{27}(q^3)_\infty^{52}(q^9)_\infty^{57} + q^{27}(q^3)_\infty^{196}(q^9)_\infty^9 + q^{29}(q^3)_\infty^{14}(q^9)_\infty^{75}$$
$$+ q^{29}(q^3)_\infty^{50}(q^9)_\infty^{63} + q^{29}(q^3)_\infty^{194}(q^9)_\infty^{15} + q^{30}(q^3)_\infty^{13}(q^9)_\infty^{78} + q^{30}(q^3)_\infty^{49}(q^9)_\infty^{66} + q^{30}(q^3)_\infty^{198}(q^9)_\infty^{18}$$
$$+ q^{31}(q^3)_\infty^3(q^9)_\infty^{84} + q^{31}(q^3)_\infty^{12}(q^9)_\infty^{81} + q^{31}(q^3)_\infty^{48}(q^9)_\infty^{69} + q^{31}(q^3)_\infty^{192}(q^9)_\infty^{21} + q^{31}(q^3)_\infty^{255}.$$

If we extract those terms in which the power is 1 (mod 3), divide by q and replace q^3 by q, we find

$$\sum_{n\geq 0} a_{16}(4782969n + 2989345)q^n$$
$$\equiv q^{10}(q)_\infty^3(q^3)_\infty^{84} + q^{10}(q)_\infty^{12}(q^3)_\infty^{81} + q^{10}(q)_\infty^{48}(q^3)_\infty^{69} + q^{10}(q)_\infty^{192}(q^3)_\infty^{21} + q^{10}(q)_\infty^{255}$$

$$\equiv q^{10}(q^3)_\infty^{85} + q^{15}(q^3)_\infty^{80}(q^9)_\infty^{15} + q^{27}(q^3)_\infty^{68}(q^9)_\infty^{51} + q^{30}(q^3)_\infty^{65}(q^9)_\infty^{60} + q^{31}(q^3)_\infty^{64}(q^9)_\infty^{63}$$
$$+ q^{75}(q^3)_\infty^{20}(q^9)_\infty^{195} + q^{78}(q^3)_\infty^{17}(q^9)_\infty^{204} + q^{79}(q^3)_\infty^{16}(q^9)_\infty^{207} + q^{90}(q^3)_\infty^5(q^9)_\infty^{240} + q^{91}(q^3)_\infty^4(q^9)_\infty^{243}$$
$$+ q^{94}(q^3)_\infty(q^9)_\infty^{252} + q^{95}(q^9)_\infty^{255}.$$

If we extract the one term in which the power is 2 (mod 3), divide by q^2 and replace q^3 by q, we obtain

$$\sum_{n\geq 0} a_{16}(14348907n + 12555283)q^n \equiv q^{31}(q^3)_\infty^{255}.$$

Since the only powers appearing are 1 (mod 3),

$$\sum_{n\geq 0} a_{16}(43046721n + 12555283)q^n \equiv 0 \text{ and } \sum_{n\geq 0} a_{16}(43046721n + 41253097)q^n \equiv 0.$$

Also

$$\sum_{n\geq 0} a_{16}(43046721n + 26904190)q^n \equiv q^{10}(q)_\infty^{255} \equiv \sum_{n\geq 0} a_{16}(n)q^{n+10}.$$

It follows that

$$a_{16}(43046721n + 26904190) \equiv a_{16}(n-10) \quad \text{and} \quad a_{16}(43046721n + 457371400) \equiv a_{16}(n)$$

and we are finished!

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