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CONGRUENCES FOR A RESTRICTED m–ARY PARTITION FUNCTION

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Abstract. We discuss a family of restricted m–ary partition functions $b_{m,j}(n)$, which is the number of m–ary partitions of n with at most $i+j$ copies of the part m^i allowed. We then use generating function dissections to prove the following family of congruences for $1 \le t \le m-1$ and $2 \le k \le m-t+1$:

$$b_{m,m-1}(m^{k+t}n + m^{k+t-1} + m^{k+t-2} + \cdots + m^k) \equiv 0 \pmod{2^{t-1}k}$$

1. Introduction

In this note, we define $b_{m,j}(n)$ to be the number of m–ary partitions of n with at most $i+j$ copies of the part m^i used. (Here m is assumed to be bigger than 1.) The function $b_{m,\infty}(n) := b_m(n)$ is then simply the number of m–ary partitions of n, the number of partitions of the integer n into powers of m. Various properties of this function b_m were extensively studied by Churchhouse [2], Rödseth [4], Andrews [1], and Gupta [3] in the late 1960's and early 1970's. More recently, this function has been revisited and additional congruences have been discovered [5]. The goal here is to restrict the function b_m in a combinatorially meaningful way with the hope that the restricted function retains some nice divisibility properties.

It is clear that the generating function for $b_m(n)$ is given by

$$B_m(q) = \sum_{n\ge 0} b_m(n)q^n = \prod_{i\ge 0} \frac{1}{1-q^{m^i}}.$$

We see that the generating function for our restricted partition function $b_{m,j}(n)$ can be written as

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$$\begin{aligned}
B_{m,j}(q) &= \sum_{n\geq 0} b_{m,j}(n)q^n \\
&= (1+q+q^2+\cdots+q^j)(1+q^m+q^{2\cdot m}+\cdots+q^{m\cdot(j+1)})\times \\
&\quad (1+q^{m^2}+q^{2\cdot m^2}+\cdots+q^{m^2\cdot(j+2)})\ldots \\
&= \prod_{i\geq 0}\left(\sum_{k=0}^{j+i} q^{m^i k}\right).
\end{aligned}$$

For the remainder of this paper, we focus our attention on $b_{m,m-1}(n)$. Note that, for $j < m-1$, the function $b_{m,j}(n)$ will equal 0 for certain values of n. It must, as it will only allow for j 1's to be used as parts. Hence, $b_{m,j}(n) = 0$ for all values n between $j+1$ and $m-1$, as well as many others. In contrast, $b_{m,m-1}(n) > 0$ for all $n \geq 0$. Indeed, $m-1$ is the smallest integer j which guarantees that $b_{m,j}(n)$ is positive for all nonnegative integers n. This makes the study of this specific function especially attractive.

2. The Initial Set of Congruences

We begin with an intermediate result needed to prove the desired congruences.

Theorem 1. *For all $n \geq 0$, and all m and k satisfying $1 \leq k \leq m$,*

$$b_{m,m-1}(m^k n) = b_{m,m+k-1}(n) + (k-1)b_{m,m+k-1}(n-1).$$

Proof. We prove this result via generating function dissections. We do so by induction on k.

First, we consider the case $k = 1$. Then we have

$$\begin{aligned}
&\sum_{n\geq 0} b_{m,m-1}(mn)q^{mn} \\
&\quad = \frac{1}{m}\sum_{l=0}^{m-1} B_{m,m-1}(\zeta^l q) \quad \text{where } \zeta = e^{2\pi i/m} \\
&\quad = \frac{1}{m}B_{m,m}(q^m)\left[\sum_{l=0}^{m-1} 1+\zeta^l q+\left(\zeta^l q\right)^2+\cdots+\left(\zeta^l q\right)^{m-1}\right] \\
&\quad = \frac{1}{m}B_{m,m}(q^m)\,[m]
\end{aligned}$$

$$= B_{m,m}(q^m).$$

Hence, we see that $\sum_{n\geq 0} b_{m,m-1}(mn)q^n = B_{m,m}(q)$ by replacing q^m by q above. (Recall that $B_{m,m}(q)$ is simply the generating function for the number of m–ary partitions of n with $i+m$ parts of the form m^i allowed.)

Now we assume the theorem is true for some k satisfying $1 \leq k < m$. This means we are assuming that

$$\sum_{n\geq 0} b_{m,m-1}(m^k n)q^n = (1+(k-1)q)B_{m,m+k-1}(q).$$

We then wish to prove the result is true for $k+1$. We have

$$\begin{aligned}
&\sum_{n\geq 0} b_{m,m-1}(m^{k+1}n)q^{mn}\\
&\quad = \frac{1}{m}\sum_{l=0}^{m-1}(1+(k-1)(\zeta^l q))B_{m,m+k-1}(\zeta^l q)\\
&\quad = \frac{1}{m}B_{m,m+k}(q^m)\left[\sum_{l=0}^{m-1}(1+(k-1)(\zeta^l q))\sum_{j=0}^{m+k-1}(\zeta^l q)^j\right]\\
&\quad = \frac{1}{m}B_{m,m+k}(q^m)\left[\sum_{l=0}^{m-1}\sum_{j=0}^{m+k-1}(1+(k-1)(\zeta^l q))(\zeta^l q)^j\right]\\
&\quad = \frac{1}{m}B_{m,m+k}(q^m)\left[\sum_{l=0}^{m-1}\sum_{j=0}^{m+k-1}\zeta^{lj}q^j+(k-1)q\sum_{l=0}^{m-1}\sum_{j=0}^{m+k-1}\zeta^{l(j+1)}q^j\right]\\
&\quad = \frac{1}{m}B_{m,m+k}(q^m)\left[m(q^0+q^m)+m(k-1)q(q^{m-1})\right] \quad \text{since } 1\leq k<m\\
&\quad = (1+kq^m)B_{m,m+k}(q^m) \quad \text{after simplification.}
\end{aligned}$$

Replacing q^m by q we obtain

$$\sum_{n\geq 0} b_{m,m-1}(m^{k+1}n)q^n = (1+kq)B_{m,m+k}(q). \tag{1}$$

This completes the proof of the theorem. □

We now prove one family of congruences using similar elementary techniques.

Theorem 2. *For all $n \geq 0$, and all m and k satisfying $2 \leq k \leq m$,*

$$\sum_{n\geq 0} b_{m,m-1}(m^{k+1}n + m^k)q^n = k(1+q)B_{m,m+k}(q).$$

Proof. From Theorem 1, we know

$$b_{m,m-1}(m^k n) = b_{m,m+k-1}(n) + (k-1)b_{m,m+k-1}(n-1)$$

for all $n \geq 0$, $m \geq 2$, and k satisfying $1 \leq k \leq m$. As seen in (1), the generating function equivalent of this statement is

$$\sum_{n\geq 0} b_{m,m-1}(m^k n)q^n = (1+(k-1)q)B_{m,m+k-1}(q).$$

We can now make the substitution $n \mapsto mn+1$ to yield the appropriate dissection.

$$\begin{aligned}
&\sum_{n\geq 0} b_{m,m-1}(m^{k+1}n + m^k)q^{mn+1} \\
&= \frac{1}{m}\sum_{l=0}^{m-1} \left(\zeta^{m-1}\right)^l F(\zeta^l q) \quad \text{where } F(q) = (1+(k-1)q)B_{m,m+k-1}(q) \\
&= \frac{1}{m}B_{m,m+k}(q^m)\left[\sum_{l=0}^{m-1} \left(\zeta^{m-1}\right)^l (1+(k-1)\zeta^l q) \sum_{j=0}^{m+k-1} \left(\zeta^l q\right)^j\right] \\
&= \frac{1}{m}B_{m,m+k}(q^m)\left[\sum_{l=0}^{m-1}\sum_{j=0}^{m+k-1} \zeta^{l(j-1)}q^j + (k-1)q\sum_{l=0}^{m-1}\sum_{j=0}^{m+k-1} \zeta^{lj}q^j\right] \\
&= \frac{1}{m}B_{m,m+k}(q^m)\left[m(q+q^{m+1}) + (k-1)q(m+mq^m)\right] \\
&= kq(1+q^m)B_{m,m+k}(q^m).
\end{aligned}$$

By dividing by q on both sides of this equality and then replacing q^m by q everywhere, we have

$$\sum_{n\geq 0} b_{m,m-1}(m^{k+1}n + m^k)q^n = k(1+q)B_{m,m+k}(q).$$

This is the desired result. □

We close this section by noting that this result implies

$$b_{m,m-1}(m^{k+1}n + m^k) = k(b_{m,m+k}(n) + b_{m,m+k}(n-1)),$$

a very nice recurrence result. Of course, it also implies that, for $2 \leq k \leq m$,

$$b_{m,m-1}(m^{k+1}n + m^k) \equiv 0 \pmod{k}.$$

3. The Full Family of Congruences

Interestingly enough, Theorem 2 is the basis case for a family of partition congruence results that we now prove.

Theorem 3. *For all $n \geq 0$, $m \geq 2$, and all t and k satisfying $1 \leq t \leq m-1$ and $2 \leq k \leq m-t+1$,*

$$\sum_{n \geq 0} b_{m,m-1}(m^{k+t}n + m^{k+t-1} + m^{k+t-2} + \cdots + m^k)q^n$$

$$= 2^{t-1}k(1+q)B_{m,m+k+t-1}(q).$$

Proof. We prove this result by induction on t. As noted above, the case when $t = 1$ is proven in Theorem 2. Moreover, the case when $t = 2$ is proven in a completely analogous fashion, so we omit that here.

We now assume

$$\sum_{n \geq 0} b_{m,m-1}(m^{k+t-1}n + m^{k+t-2} + m^{k+t-3} + \cdots + m^k)q^n$$

$$= 2^{t-2}k(1+q)B_{m,m+k+t-2}(q)$$

for $2 \leq t < m-1$. We wish to prove that

$$\sum_{n \geq 0} b_{m,m-1}(m^{k+t}n + m^{k+t-1} + m^{k+t-2} + \cdots + m^k)q^n$$

$$= 2^{t-1}k(1+q)B_{m,m+k+t-1}(q).$$

As in the proof of Theorem 2, we note that

$$\begin{aligned}
&\sum_{n \geq 0} b_{m,m-1}(m^{k+t}n + m^{k+t-1} + \cdots + m^k)q^{mn+1} \\
&\quad = \frac{1}{m}\sum_{l=0}^{m-1} \left(\zeta^{m-1}\right)^l F(\zeta^l q) \quad \text{where } F(q) = 2^{t-2}k(1+q)B_{m,m+k+t-2}(q) \\
&\quad = \frac{1}{m}B_{m,m+k+t-1}(q^m)\left[\sum_{l=0}^{m-1} \left(\zeta^{m-1}\right)^l \left(2^{t-2}k(1+\zeta^l q)\right) \sum_{j=0}^{m+k+t-2} \left(\zeta^l q\right)^j\right] \\
&\quad = \frac{1}{m}(2^{t-2}k)B_{m,m+k+t-1}(q^m)\times
\end{aligned}$$

$$\left[\sum_{l=0}^{m-1}\sum_{j=0}^{m+k+t-2}\zeta^{l(j-1)}q^j + q\sum_{l=0}^{m-1}\sum_{j=0}^{m+k+t-2}\zeta^{lj}q^j\right]$$
$$= \frac{1}{m}(2^{t-2}k)B_{m,m+k+t-1}(q^m)\left[m(q+q^{m+1}) + q(m+mq^m)\right]$$
$$= 2^{t-1}kq(1+q^m)B_{m,m+k+t-1}(q^m).$$

By dividing by q on both sides of this equality and then replacing q^m by q everywhere, we have

$$\sum_{n\geq 0} b_{m,m-1}(m^{k+t}n + m^{k+t-1} + m^{k+t-2} + \cdots + m^k)q^n$$
$$= 2^{t-1}k(1+q)B_{m,m+k+t-1}(q). \qquad \square$$

We close with a few remarks. First, Theorem 3 obviously yields many nice congruence properties. Namely, for all $n \geq 0$, $m \geq 2$, and all t and k satisfying $1 \leq t \leq m-1$ and $2 \leq k \leq m-t+1$,

$$b_{m,m-1}(m^{k+t}n + m^{k+t-1} + m^{k+t-2} + \cdots + m^k)q^n \equiv 0 \pmod{2^{t-1}k}.$$

Secondly, the key in these results is the restriction placed on k so that the generating functions do not become overly complicated. Indeed, by restricting k in the manner above, the sums involved are quite small and rather clean.

Finally, we have focused our attention on iteratively applying the transformation $n \mapsto mn+1$ many times. It should be noted that analogous results also hold when one applies $n \mapsto mn+2$ in place of the mapping $n \mapsto mn+1$. The proofs are synonymous with those above, so we omit them here.

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