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SOME RELATIONS FOR PARTITIONS INTO FOUR SQUARES

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1. Introduction

Let $p_{4\square}(n)$ denote the number of partitions of n into four squares (of non-negative integers). In an earlier paper [1] it was shown that

$$\sum_{n\geq 0} p_{4\square}(n)q^n = \frac{1}{384}\Big(\phi(q)^4 + 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) + 18\phi(q)^2 + 24\phi(q)\phi(q^2) + 12\phi(q^2)^2$$
$$+ 32\phi(q)\phi(q^3) + 60\phi(q) + 36\phi(q^2) + 32\phi(q^3) + 48\phi(q^4) + 105\Big)$$

where

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2}.$$

We expanded this generating function to the 2000^{th} power, and discovered that $p_{4\square}(n)$ satisfies a few, a very few, simple arithmetic relations.
By contrast, as is well-known, $r_4(n)$, the number of representations of n as a sum of four squares, exhibits many arithmetic properties, namely,
if n is 2–free then

$$r_4(2^\alpha n) = 3r_4(n) \text{ if } \alpha \geq 1,$$

and if p is an odd prime and n is p–free then

$$r_4(p^\alpha n) = \frac{p^{\alpha+1}-1}{p-1} r_4(n).$$

In this paper we shall show that

$$p_{4\Box}(8n) = p_{4\Box}(2n) \tag{1}$$

and

$$p_{4\Box}(32n+28) = 3p_{4\Box}(8n+7). \tag{2}$$

If we combine these, we find

$$p_{4\Box}(4^\alpha(8n+7)) = 3p_{4\Box}(8n+7) \quad \text{for } \alpha \geq 1. \tag{3}$$

We shall also show that

$$p_{4\Box}(72n+69) \text{ is even.} \tag{4}$$

2. Proof of (1)

Suppose

$$2n = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Then

$$8n = (2m_1)^2 + (2m_2)^2 + (2m_3)^2 + (2m_4)^2.$$

Conversely, suppose

$$8n = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

By considering this modulo 8, since squares are congruent to 0, 1 or 4 (mod 8), we see that $x_1,\ x_2,\ x_3,\ x_4$ are all even. Let $m_1 = x_1/2$, $m_2 = x_2/2$, $m_3 = x_2/2$, $m_4 = x_4/2$. Then

$$2n = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Thus there is a one–to–one correspondence between partitions of $2n$ into four squares and partitions of $8n$ into four squares. This proves (1). ■

3. Proof of (2)

Suppose
$$8n+7 = m_1^2+m_2^2+m_3^2+m_4^2$$
with $m_1,\ m_2,\ m_3,\ m_4 \geq 0$. Then w.l.o.g.
$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ \pm1 \\ \pm1 \\ \pm1 \end{pmatrix} \pmod 4$$
and $m_2 \geq m_3 \geq m_4 > 0$.

Then $x_1,\ x_2,\ x_3,\ x_4$ are uniquely defined by
$$\{x_1,\ x_2,\ x_3,\ x_4\} = \{m_1,\ \pm m_2,\ \pm m_3,\ \pm m_4\},$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \pmod 4$$
and
$$x_2 \geq x_3 \geq x_4.$$
Let
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1+x_2+x_3+x_4 \\ x_1+x_2-x_3-x_4 \\ x_1-x_2+x_3-x_4 \\ x_1-x_2-x_3+x_4 \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} -x_1+x_2+x_3+x_4 \\ -x_1+x_2-x_3-x_4 \\ -x_1-x_2+x_3-x_4 \\ -x_1-x_2-x_2+x_4 \end{pmatrix}.$$
Then
$$(2m_1)^2+(2m_2)^2+(2m_3)^2+(2m_4)^2 = y_1^2+y_2^2+y_3^2+y_4^2 = z_1^2+z_2^2+z_3^2+z_4^2 = 32n+28,$$
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \equiv \begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \end{pmatrix} \pmod 8, \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \\ 5 \\ 5 \end{pmatrix} \pmod 8 \text{ or } \textit{vice versa}$$
and
$$y_2 \geq y_3 \geq y_4, \quad z_2 \geq z_3 \geq z_4.$$

Conversely, if

$$32n + 28 = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

with $m_1,\ m_2,\ m_3,\ m_4 \geq 0$, then w.l.o.g.

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ \pm 2 \\ \pm 2 \\ \pm 2 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 5 \\ \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 1 \\ \pm 5 \\ \pm 5 \\ \pm 5 \end{pmatrix} \pmod 8$$

and $m_2 \geq m_3 \geq m_4 > 0$.

In the first case, define $x_1,\ x_2,\ x_3,\ x_4$ by $x_1 = m_1/2$, $x_2 = m_2/2$, $x_3 = m_3/2$, $x_4 = m_4/2$ and then

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 8n + 7.$$

Otherwise, $x_1,\ x_2,\ x_3,\ x_4$ are uniquely defined by

$$\{x_1,\ x_2,\ x_3,\ x_4\} = \{\pm m_1,\ \pm m_2,\ \pm m_3,\ \pm m_4\},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \equiv \begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 5 \\ 5 \\ 5 \end{pmatrix} \pmod 8$$

and $x_2 \geq x_3 \geq x_4$.

Let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 + x_3 + x_4)/4 \\ (x_1 + x_2 - x_3 - x_4)/4 \\ (x_1 - x_2 + x_3 - x_4)/4 \\ (x_1 - x_2 - x_3 + x_4)/4 \end{pmatrix}.$$

Then

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \pmod 4,$$

$y_2 \geq y_3 \geq y_4$ and

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 8n + 7.$$

This establishes a one–to–three correspondence between partitions of $8n + 7$ into four squares and partitions of $32n + 28$ into four squares. This proves (2). ■

An example

If $n = 7$, $8n + 7 = 63$, $32n + 28 = 252$,

$$63 = 7^2 + 3^2 + 2^2 + 1^2 = 6^2 + 5^2 + 1^2 + 1^2 = 6^2 + 3^2 + 3^2 + 3^2 = 5^2 + 5^2 + 3^2 + 2^2.$$

These partitions correspond to the vectors

$$\begin{pmatrix} 2 \\ 1 \\ -3 \\ -7 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 5 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ -3 \\ -3 \\ -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 5 \\ 5 \\ -3 \end{pmatrix}.$$

As described above, these yield the following twelve vectors corresponding to partitions of 252 into four squares,

$$\left\{ \begin{pmatrix} 4 \\ 2 \\ -6 \\ -14 \end{pmatrix}, \begin{pmatrix} -7 \\ 13 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -11 \\ 9 \\ 1 \\ -7 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 12 \\ 10 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 13 \\ 9 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -11 \\ -11 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 12 \\ -6 \\ -6 \\ -6 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \\ 9 \\ 9 \end{pmatrix}, \begin{pmatrix} -15 \\ -3 \\ -3 \\ -3 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 4 \\ 10 \\ 10 \\ -6 \end{pmatrix}, \begin{pmatrix} 9 \\ 5 \\ 5 \\ -11 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ -15 \end{pmatrix} \right\}.$$

Thus we find

$$\begin{aligned} 252 &= 15^2 + 5^2 + 1^2 + 1^2 = 15^2 + 3^2 + 3^2 + 3^2 = 14^2 + 6^2 + 4^2 + 2^2 = 13^2 + 9^2 + 1^2 + 1^2 \\ &= 13^2 + 7^2 + 5^2 + 3^2 = 12^2 + 10^2 + 2^2 + 2^2 = 12^2 + 6^2 + 6^2 + 6^2 = 11^2 + 11^2 + 3^2 + 1^2 \\ &= 11^2 + 9^2 + 7^2 + 1^2 = 11^2 + 9^2 + 5^2 + 5^2 = 10^2 + 10^2 + 6^2 + 4^2 = 9^2 + 9^2 + 9^2 + 3^2. \end{aligned}$$

4. Proof of (4)

Suppose

$$72n + 69 = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

with $m_1,\ m_2,\ m_3,\ m_4 \geq 0$.

It is easy to show by considering this equation modulo 9 that precisely one of the m_i is congruent to 0 modulo 3, so

$$72n + 69 - 9m^2 = m_1^2 + m_2^2 + m_3^2.$$

We shall show that for each m, the number of solutions of this equation is even.

W.l.o.g.

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 2 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 2 \\ \pm 2 \\ \pm 4 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 4 \\ \pm 4 \\ \pm 1 \end{pmatrix} \pmod 9$$

and $m_1 \geq m_2$.

Then $x_1,\ x_2,\ x_3$ are uniquely defined by

$$\{x_1,\ x_2,\ x_3\} = \{\pm m_1,\ \pm m_2,\ \pm m_3\},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \pmod 9$$

and

$$x_1 \geq x_2.$$

Let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (x_1 - 2x_2 - 2x_3)/3 \\ (-2x_1 + x_2 - 2x_3)/3 \\ (-2x_1 - 2x_2 + x_3)/3 \end{pmatrix}.$$

Then

$$y_1^2 + y_2^2 + y_3^2 = x_1^2 + x_2^2 + x_3^2 = 72n + 69 - 9m^2,$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \pmod 9$$

and

$$y_1 \geq y_2.$$

Indeed, $y_1 - y_2 = x_1 - x_2$, $\ y_2 - y_3 = x_2 - x_3$ and $y_1 + y_2 + y_3 = -(x_1 + x_2 + x_3)$!

The transformation $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is an involution with no fixed points. For fixed points have $x_1 + x_2 + x_3 = 0$, or, $x_3 = -x_1 - x_2$, and then

$$x_1^2 + x_2^2 + x_3^2 = 2(x_1^2 + x_1 x_2 + x_2^2).$$

But the highest power of 2 that divides $2(x_1^2 + x_1 x_2 + x_2^2)$ is odd, while the highest power of 2 that divides $72n + 69 - 9m^2$ is even.

So the number of solutions of

$$72n + 69 - 9m^2 = m_1^2 + m_2^2 + m_3^2$$

with $m_1,\ m_2,\ m_3 \geq 0$ is even. This proves (4). ■

An example

If $n = 3$, $72n + 69 = 285$,

$$\begin{aligned}285 &= 16^2 + 5^2 + 2^2 + 0^2 = 16^2 + 4^2 + 3^2 + 2^2 = 14^2 + 9^2 + 2^2 + 2^2 = 14^2 + 8^2 + 5^2 + 0^2 \\ &= 14^2 + 8^2 + 4^2 + 3^2 = 14^2 + 7^2 + 6^2 + 2^2 = 13^2 + 10^2 + 4^2 + 0^2 = 13^2 + 8^2 + 6^2 + 4^2 \\ &= 12^2 + 11^2 + 4^2 + 2^2 = 12^2 + 10^2 + 5^2 + 4^2 = 11^2 + 10^2 + 8^2 + 0^2 = 11^2 + 8^2 + 8^2 + 6^2 \\ &= 10^2 + 10^2 + 9^2 + 2^2 = 10^2 + 10^2 + 7^2 + 6^2.\end{aligned}$$

These partitions correspond to the following vectors, arranged in corresponding pairs.

$$(m=0)\ \left\{\begin{pmatrix}16\\-2\\-5\end{pmatrix}, \begin{pmatrix}10\\-8\\-11\end{pmatrix}\right\}, \left\{\begin{pmatrix}-5\\-14\\-8\end{pmatrix}, \begin{pmatrix}13\\4\\10\end{pmatrix}\right\},\ (m=1)\ \left\{\begin{pmatrix}16\\-2\\4\end{pmatrix}, \begin{pmatrix}4\\-14\\-8\end{pmatrix}\right\},$$

$$(m=2)\ \left\{\begin{pmatrix}7\\-2\\-14\end{pmatrix}, \begin{pmatrix}13\\4\\-8\end{pmatrix}\right\}, \left\{\begin{pmatrix}-8\\-8\\-11\end{pmatrix}, \begin{pmatrix}10\\10\\7\end{pmatrix}\right\},\ (m=3)\ \left\{\begin{pmatrix}-2\\-2\\-14\end{pmatrix}, \begin{pmatrix}10\\10\\-2\end{pmatrix}\right\},$$

$$(m=4)\ \left\{\begin{pmatrix}-2\\-11\\4\end{pmatrix}, \begin{pmatrix}4\\-5\\10\end{pmatrix}\right\}.$$

Reference

[1] M. D. Hirschhorn, Some formulae for partitions into squares, Discrete Math. (to appear).