Abstract

In this paper, we consider sequences comprised of n $(m-1)$'s and r -1's (where $m \geq 2$) with the sum of each subsequence of the first j terms nonnegative. We will denote the number of such sequences as $\left\{ {n \atop r} \right\}_{m-1}$. Our goal is to present various results involving $\left\{ {n \atop r} \right\}_{m-1}$, including an interpretation of the sequences counted by $\left\{ {n \atop r} \right\}_{m-1}$ which truly generalizes the proof that $C_n = \frac{1}{n+1}\binom{2n}{n}$. In particular, we pay special attention to the case $r = (m-1)n$ (the largest allowable value of r for fixed m) and prove that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}.$$

Generalizing Bailey's Generalization of the Catalan Numbers *

Darrin D. Frey and James A. Sellers
Department of Science and Mathematics
Cedarville College
P.O. Box 601
Cedarville, OH 45314

May 31, 2001

Introduction

In a recent note in the *Mathematics Magazine* [1], D. F. Bailey developed a formula for the number of sequences

$$a_1, a_2, \ldots, a_{n+r}$$

comprised of n 1's and r -1's such that $\sum_{i=1}^{j} a_j \geq 0$ for each $j = 1, 2, \ldots, n+r$.

He denoted the number of such sequences as $\left\{ {n \atop r} \right\}$ and noted that

$$\left\{ {n \atop r} \right\} = \left\{ {n \atop r-1} \right\} + \left\{ {n-1 \atop r} \right\} \tag{1}$$

* *Fibonacci Quarterly* **39**, no. 2 (May 2001), 142-148

for $1 < r < n$ and

$$\left\{ {n \atop n} \right\} = \left\{ {n \atop n-1} \right\} \tag{2}$$

for each $n \geq 1$. From these facts, it is easy to build the following table of values for $\left\{ {n \atop r} \right\}$.

$n\backslash r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
8	1	8	35	110	275	572	1001	1430	1430

Table 1: Values for $\left\{ {n \atop r} \right\}$

Indeed, Bailey obtains the identity

$$\left\{ {n \atop r} \right\} = \frac{(n+1-r)(n+2)(n+3)\dots(n+r)}{r!}$$

whenever $n \geq r \geq 2$. Bailey closes the paper by noting that

$$\left\{ {n \atop n} \right\} = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number C_n. This is not surprising, as one of the classical combinatorial interpretations of C_n is the number of sequences of n 1's and n -1's that satisfy the subsequence sum restriction mentioned above. In this context, Bailey provided a very nice generalization of the classical Catalan numbers.

In this paper, we generalize Bailey's work by considering sequences comprised of n $(m-1)$'s and r -1's (where $m \geq 2$) with the sum of each subsequence of the first j terms nonnegative. We will denote the number of such sequences as $\left\{ {n \atop r} \right\}_{m-1}$.

It is clear that the recurrences similar to (1) and (2) are satisfied for general values of m. Namely,

$$\left\{ {n \atop r} \right\}_{m-1} = \left\{ {n-1 \atop r} \right\}_{m-1} + \left\{ {n \atop r-1} \right\}_{m-1} \tag{3}$$

for $1 < r < n$ and

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \left\{ {n \atop (m-1)n-1} \right\}_{m-1} = \left\{ {n \atop (m-1)n-2} \right\}_{m-1} = \dots$$

$$\dots = \left\{ {n \atop (m-1)n-(m-1)} \right\}_{m-1} \tag{4}$$

for each $n \geq 1$. The recurrence in (3) is seen in a straightforward manner. Take a sequence of n $(m-1)$'s and r -1's which is counted by $\left\{ {n \atop r} \right\}_{m-1}$. The last element in the sequence is either an $m-1$ or a -1. If it is an $m-1$, then the preceding subsequence is one of those counted by $\left\{ {n-1 \atop r} \right\}_{m-1}$. On the other hand, if the last element of our original sequence is a -1, then the preceding subsequence is one of those enumerated in $\left\{ {n \atop r-1} \right\}_{m-1}$.

To establish (4), consider a sequence of n $(m-1)$'s and $(m-1)n$ -1's counted by $\left\{ {n \atop (m-1)n} \right\}_{m-1}$. It must be the case that the last element in this sequence is -1. (If not, then one of the subsequence sums would have to be negative, which is contradictory.) Hence, the subsequence preceding this final -1 will also satisfy the property that all of its subsequence sums are positive. Therefore, this preceding subsequence will be enumerated by $\left\{ {n \atop (m-1)n-1} \right\}_{m-1}$. In more general terms, it is clear that the last $m-1$ elements of our sequence of n $(m-1)$'s and $n(m-1)$ -1's must be -1. Thus, the same argument as that above can be used to prove the full set of equalities in (4).

We include here a table comparable to Table 1 above in the case of $m = 3$.

$n \backslash r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1	1						
2	1	2	3	3	3				
3	1	3	6	9	12	12	12		
4	1	4	10	19	31	43	55	55	55
5	1	5	15	34	65	108	163	218	273
6	1	6	21	55	120	228	391	609	882
7	1	7	28	83	203	431	822	1431	2313
8	1	8	36	119	322	753	1575	3006	5319

Table 2: Values for $\left\{ {n \atop r} \right\}_2$

Our goal is to present various results involving $\left\{ {n \atop r} \right\}_{m-1}$, including an interpretation of the sequences counted by $\left\{ {n \atop r} \right\}_{m-1}$ which truly generalizes the proof that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

which appears in [2]. In particular, we pay special attention to the case when $r = (m-1)n$ (the largest allowable value of r for fixed m) and prove that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{5}$$

This is a clear generalization of C_n. Indeed, these quantities also enjoy a rich history and background, and can be found in the works of Raney [6] and others. See [5] and [7] for additional discussion.

The Main Generalization of the Catalan Numbers

We now consider sequences of the form

$$a_1, a_2, \ldots, a_{n+r}$$

containing n $(m-1)$'s and r -1's for a fixed value of m larger than 1. Note that the total number of such sequences (with no restrictions) is $\binom{n+r}{n}$. We wish to count, in a natural way, the number of such sequences which satisfy

$$a_1 + a_2 + \ldots + a_j \geq 0 \tag{6}$$

for each $j = 1, 2, \ldots, n+r$. We will do so by subtracting from $\binom{n+r}{n}$ the number of sequences $\left\langle {n \atop r} \right\rangle_{m-1}$ which violate (6) above for at least one value of j. (These sequences will be referred to affectionately as "bad.") This use of inclusion/exclusion is, in essence, the approach taken in [2] to prove the formula for C_n, although the proof in [2] is not directly generalizable.

For simplicity, we first focus our attention on the case $r = (m-1)n$. We pick up the more general case in the next section.

On our way to a closed form for $\left\{ {n \atop (m-1)n} \right\}_{m-1}$, we first state a lemma.

Lemma 1.

$$\sum_{k=0}^{n} \frac{w}{w+dk} \binom{p-bk}{n-k} \binom{q+bk}{k} = \binom{p+q}{n}$$
$$+ \sum_{k=1}^{n} \binom{p+q-k}{n-k} \frac{(wb-qd)(wb-(q-1)d)\cdots(wb-(q-k+1)d)}{(w+d)(w+2d)\cdots(w+kd)}$$

for all values of p, q, w, n, b and d for which the terms are defined.

Proof. This result is proven by H. Gould and J. Kaucky in [4]. $\square$

Using Lemma 1, we can prove the following corollary.

Corollary 1. *For all $m \geq 2$,*

$$\sum_{k=0}^{n-1} \frac{1}{(m-1)k+1} \binom{mk}{k} \binom{mn-mk-1}{n-k} = (m-1)\binom{mn}{n-1}.$$

Proof. We apply Lemma 1 with $b = m, p = mn-1, q = 0, w = 1$ and $d = m-1$. Then we have, following [4],

$$\sum_{k=0}^{n} \frac{1}{1+(m-1)k} \binom{mn-1-mk}{n-k} \binom{mk}{k}$$

$$
\begin{aligned}
&= \binom{mn-1}{n} + \sum_{k=1}^{n} \binom{mn-1-k}{n-k} \\
&= \sum_{k=0}^{n} \binom{mn-1-k}{n-k} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{-mn+k+1+n-k-1}{n-k} \quad \text{using (3.4), [4]} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{-(m-1)n}{n-k} \\
&= \sum_{j=n}^{0} (-1)^{j} \binom{-(m-1)n}{j} \quad \text{using } j = n-k \\
&= \sum_{j=0}^{n} (-1)^{j} \binom{-(m-1)n}{j} \\
&= \binom{n+(m-1)n}{n} \quad \text{using identity 1.49, [3]} \\
&= \binom{mn}{n}.
\end{aligned}
$$

Thus, we know that

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \frac{1}{(m-1)k+1} \binom{mk}{k} \binom{m(n-k)-1}{n-k} \\
&= \binom{mn}{n} - \frac{1}{(m-1)n+1} \binom{mn}{n} \binom{-1}{0} \\
&= \binom{mn}{n} - \frac{1}{(m-1)n+1} \binom{mn}{n} \cdot 1 \\
&= \frac{(m-1)n}{(m-1)n+1} \binom{mn}{n} \\
&= (m-1) \binom{mn}{n-1}.
\end{aligned}
$$

$\square$

We are now in position to state the main result for $\left\{ {n \atop (m-1)n} \right\}_{m-1}$.

Theorem 1. *For all $n \geq 1$ and $m \geq 2$,*

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1} \binom{mn}{n}.$$

Proof. We will prove this theorem by induction on n. First, we recall that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \binom{n+(m-1)n}{n} - \left\langle {n \atop (m-1)n} \right\rangle_{m-1}$$

and we focus on the "bad" sequences counted by $\left\langle {n \atop (m-1)n} \right\rangle_{m-1}$. Let

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_{mn} \rangle$$

be a bad sequence of n $(m-1)$'s and $(m-1)n$ -1's and let j be the first subscript for which the partial sum $S_j = \sum_{i=1}^{j} a_i < 0$. (The existence of j is guaranteed since $\mathbf{a}$ is a bad sequence.) Then $S_j = -1$ and $a_j = -1$ (by the minimality of j) so that $S_{j-1} = 0$. Therefore $j \equiv 1 \pmod{m}$.

We now set $k = \frac{j-1}{m}$. Then there are k $(m-1)$'s in the partial sequence $\mathbf{a}_{j-1} = \langle a_1, \ldots, a_{j-1} \rangle$, so $\mathbf{a}_{j-1}$ is a "good" sequence with k $(m-1)$'s and $(m-1)k$ -1's by the minimality of k. Moreover, the subsequence

$$\hat{\mathbf{a}}_j = \langle a_{j+1}, a_{j+2}, \ldots a_{mn} \rangle$$

is an arbitrary sequence of $(n-k)$ $(m-1)$'s and $((m-1)(n-k)-1)$ -1's. (Remember that $a_j = -1$.) Since there are $\left\{ {k \atop (m-1)k} \right\}_{m-1}$ ways to choose a good sequence with k $(m-1)$'s and $(m-1)k$ -1's and $\binom{mn-mk-1}{n-k}$ ways to choose an arbitrary sequence of $(n-k)$ $(m-1)$'s and $((m-1)(n-k)-1)$ -1's, then the number of bad sequences with n $(m-1)$'s and $(m-1)n$ -1's with the first bad partial sequence having length $mk+1$ is $\left\{ {k \atop (m-1)k} \right\}_{m-1} \binom{mn-mk-1}{n-k}$. Hence

$$\left\langle {n \atop (m-1)n} \right\rangle_{m-1} = \sum_{k=0}^{n-1} \left\{ {k \atop (m-1)k} \right\}_{m-1} \binom{mn-mk-1}{n-k}. \qquad (7)$$

A quick comment is in order regarding the index of summation in (7). Since $j \equiv 1 \bmod m$, the smallest possible value of j is 1 whence $k=0$ is the smallest value of k (since $k = \frac{j-1}{m}$). Since the whole sequence has length mn, the largest value of j is $mn-m+1$ and hence the largest value of k is $\frac{mn-m+1-1}{m} = n-1$.

Now by induction,

$$\left\{ {k \atop (m-1)k} \right\}_{m-1} = \frac{1}{(m-1)k+1}\binom{mk}{k},$$

so

$$\left\langle {n \atop (m-1)n} \right\rangle_{m-1} = \sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{mn-mk-1}{n-k}.$$

Now Corollary 1 can be applied.

$$\begin{aligned} \left\langle {n \atop (m-1)n} \right\rangle_{m-1} &= \sum_{k=0}^{n-1} \frac{1}{(m-1)k+1}\binom{mk}{k}\binom{mn-mk-1}{n-k} \\ &= (m-1)\binom{mn}{n-1}. \end{aligned}$$

Our result is now in reach.

$$\begin{aligned}
\left\{ {n \atop (m-1)n} \right\}_{m-1} &= \binom{n+(m-1)n}{n} - \left\langle {n \atop (m-1)n} \right\rangle_{m-1} \\
&= \binom{mn}{n} - (m-1)\binom{mn}{n-1} \\
&= \binom{mn}{n} - (m-1)\cdot\frac{n}{(mn-n+1)}\binom{mn}{n} \\
&= \left[1-\frac{n(m-1)}{mn-n+1}\right]\binom{mn}{n} \\
&= \frac{1}{(m-1)n+1}\binom{mn}{n} \quad \text{after simplification.}
\end{aligned}$$

Therefore, we see that

$$\left\{ {n \atop (m-1)n} \right\}_{m-1} = \frac{1}{(m-1)n+1}\binom{mn}{n}.$$

$\square$

As noted in the introductory section, the values $\frac{1}{(m-1)n+1}\binom{mn}{n}$ have appeared in the past. However, we are unaware of their interpretation as the number of sequences described above. Moreover, the proof technique utilized in Theorem 1 does not seem readily available in the literature.

The General Case

Our initial motivation in this study (in the spirit of Bailey) was to find a closed formula for $\left\{ {n \atop r} \right\}_{m-1}$ for all r satisfying $1 \le r \le (m-1)n$. The completion of this task has proven elusive. However, we can generalize (7) above to determine $\left\langle {n \atop r} \right\rangle_{m-1}$. Then it is clear that

$$\left\{ {n \atop r} \right\}_{m-1} = \binom{n+r}{n} - \left\langle {n \atop r} \right\rangle_{m-1}.$$

We now look at the generalization of (7).

Theorem 2.

$$\left\langle {n \atop r} \right\rangle_{m-1} = \sum_{k=0}^{\lceil \frac{r}{m-1} \rceil - 1} \frac{1}{(m-1)k+1} \binom{mk}{k} \binom{n+r-mk-1}{n-k} \tag{8}$$

Proof. The proof of this is essentially the same as that in Theorem 1. The major difference is that the index of summation must be modified. To determine the extreme values of k we analyze as before. Since $j \equiv 1 \bmod m$, the smallest possible value of j is 1 whence $k = 0$ is the smallest value of k (since $k = \frac{j-1}{m}$). Now, any "good" subsequence of $\mathbf{a}$ has km terms and so will have $(km - k)$ -1's in it. But if $km - k \geq r$, there are no -1's left to make the sequence bad. That is, we may not have $k \geq \frac{r}{m-1}$, so the maximum value of k is $\lfloor \frac{r}{m-1} \rfloor$ if r is not an integer multiple of $m - 1$, and $\frac{r}{m-1} - 1$ if r is an integer multiple of $m - 1$. A more efficient way of expressing the maximum value of k is as $\lceil \frac{r}{m-1} \rceil - 1$. □

Unfortunately, we have been unable to determine a closed formula for (8). However, we note that this is still a useful insight, at least in a computational sense. Indeed, if one wants to determine (for example) $\left\{ {100 \atop 44} \right\}_4$, with $m = 5$, $n = 100$, and $r = 44$, then (8) provides a very feasible way to calculate $\left\langle {100 \atop 44} \right\rangle_4$, so that

$$\left\{ {100 \atop 44} \right\}_4 = \binom{144}{44} - \left\langle {100 \atop 44} \right\rangle_4 .$$

In this example, the sum in (8) only contains $\lceil \frac{44}{4} \rceil$ or 11 terms, each of which is simply a weighted product of two binomial coefficients. This is much quicker than calculating $\left\{ {100 \atop 44} \right\}_4$ from the recurrences (3) and (4).

Concluding Thoughts and Questions

While we have not fully reached our initial goal, we are satisfied with the results obtained, especially since the approach seems quite novel. We now share two thoughts in closing.

First, we covet a closed formula for the sum in (8). It is unclear how to accomplish this task. Second, we note a fairly interesting residual result from Bailey's work. Bailey proved that

$$\left\{ {n \atop r} \right\}_1 = \frac{(n+1-r)(n+r)!}{(n+1)n!r!} = \frac{n+1-r}{n+1}\binom{n+r}{n}.$$

Thus, it is clear that

$$\begin{aligned} \left\langle {n \atop r} \right\rangle_1 &= \binom{n+r}{r} - \frac{n+1-r}{n+1}\binom{n+r}{r} \\ &= \left(1 - \frac{n+1-r}{n+1}\right)\binom{n+r}{r} \end{aligned}$$

$$= \quad \frac{r}{n+1}\binom{n+r}{r}.$$

By (8) above, we then have

$$\sum_{k=0}^{r-1} \frac{1}{k+1}\binom{2k}{k}\binom{n+r-2k-1}{n-k} = \frac{r}{n+1}\binom{n+r}{r}.$$

The proof of this summation result does not appear to be within reach via known tools such as Lemma 1, and we have been unable to prove this identity directly. A direct combinatorial proof of this result would be nice to see. If found, such a proof might allow us to better see a closed formula for the sum in (8).

Acknowledgements

The authors gratefully acknowledge Henry Gould for his invaluable assistance.

References

[1] D. F. Bailey, Counting Arrangements of 1's and −1's, *Mathematics Magazine*, **69**, (2) (1996), 128-131.

[2] R. A. Brualdi, "Introductory Combinatorics", 2nd Edition, Prentice Hall, Englewood Cliffs, NY, 1992.

[3] H. W. Gould, "Combinatorial Identities; a standardized set of tables listing 500 binomial coefficient summations", West Virginia University Press, Morgantown, WV, 1972.

[4] H. W. Gould and J. Kaucky, Evaluation of a Class of Binomial Coefficient Summations *Journal of Combinatorial Theory*, **1** (2) (1966), 233-247.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, "Concrete Mathematics", 2nd Edition, Addison-Wesley, Reading, Massachusetts, 1994.

[6] G. N. Raney, Functional composition patterns and power series reversal, *Trans. Amer. Math. Soc.*, **94** (1960), 441-451.

[7] R. Stanley, "Enumerative Combinatorics", Wadsworth and Brooks/Cole, Monterey, California, 1986.

1991 *Mathematics Subject Classification.* 05A10