Computational Proofs of Congruences for 2–Colored Frobenius Partitions

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Abstract

In 1994, the following infinite family of congruences was conjectured for the partition function $c\phi_2(n)$ which counts the number of 2–colored Frobenius partitions of n:

For all $n \geq 0$ and $\alpha \geq 1$,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha},$$

where λ_α is the least positive reciprocal of 12 modulo 5^α.

In this paper, the first four cases of this family are proven.

1 Background and Introduction

In his 1984 Memoir of the American Mathematical Society, George E. Andrews [2] introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this note, we will focus our attention on one of these functions, namely $c\phi_2(m)$, which denotes the number of generalized Frobenius partitions of m with 2 colors. In [2], Andrews

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gives the generating function for $c\phi_2(m)$:

$$\sum_{m\geq 0} c\phi_2(m)q^m = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4 (q^4;q^4)_\infty}, \tag{1}$$

where $(a;b)_\infty = (1-a)(1-ab)(1-ab^2)(1-ab^3)\ldots$. Andrews then proves the following: For all $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod 5, \quad \text{and} \tag{2}$$

$$c\phi_2(2n+1) \equiv 0 \pmod 4. \tag{3}$$

More recently, Sellers [9] conjectured the following infinite family of congruences satisfied by $c\phi_2$:

Conjecture 1. *For all $n \geq 0$ and $\alpha \geq 1$,*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{4}$$

where λ_α is the least positive reciprocal of 12 modulo 5^α.

The case $\alpha = 1$ is (2) mentioned above.

The reader will note the similarity of this conjecture to the well–known family of congruences for $p(m)$, the classical partition function of m: For all $n \geq 0$,

$$p(5^\alpha n + \gamma_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{5}$$

where γ_α is the least positive reciprocal of 24 modulo 5^α. (For two different proofs of (5), see [1] and [6].) Unfortunately, (4) has proven to be much more difficult to prove than (5).

The goal of this note is to prove the following:

Theorem 1. *For all $n \geq 0$ and $\alpha = 1, 2, 3, 4$,*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{6}$$

where λ_α is the least positive reciprocal of 12 modulo 5^α.

In order to prove this theorem, we implement a finitization technique developed recently. (See, for example, [3].) In essence, we prove that, for fixed α,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \text{ for all } n$$

if and only if

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \text{ for all } n \leq C(\alpha)$$

where $C(\alpha)$ is an explicit constant dependent on α. We will then compute all values of $c\phi_2$ needed to utilize the equivalence above. The development of $C(\alpha)$ will require the theory of modular forms as outlined below.

2 Determination of $C(\alpha)$

In this section, we use the theory of modular forms to determine the constant $C(\alpha)$. We will do so by constructing a modular form whose Fourier coefficients inherit the congruence properties modulo 5^α of $c\phi_2$ in the desired arithmetic progression. Then, thanks to a theorem of Sturm [10], we will be able to provide explicitly a constant $C(\alpha)$ such that if a congruence for the Fourier coefficients of our modular form (or equivalently, for $c\phi_2$) holds for all $n \leq C(\alpha)$, the congruence must hold for all n.

For a general introduction to the theory of modular forms, see [7]. For an exposition focused on the results we use below, see [3].

We now state Sturm's Theorem.

Theorem 2 (Sturm). *If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=0}^{\infty} b(n)q^n$ are holomorphic modular forms of weight k with respect to some congruence subgroup Γ of $SL_2(\mathbb{Z})$ with integer coefficients, then $f(z) \equiv g(z) \pmod{l}$ where l is prime if and only if*

$$\mathrm{Ord}_l(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma].$$

where $\mathrm{Ord}_l(F(q)) := \min\{n \mid A(n) \not\equiv 0 \pmod{l}\}$.

Sturm's Theorem also holds when the prime l is replaced by 5^α, or in fact by any positive integer. Thus, when we let $g(z) = 0$, Sturm's Theorem allows us to determine when the coefficients $a(n)$ of a holomorphic modular form have the property that $a(n) \equiv 0 \pmod{5^\alpha}$ for all n.

We are now ready to state the main result needed to prove Theorem 1 above.

Theorem 3. *Suppose α is a positive integer, and let*

$$C(\alpha) := 6(b - 1 + 4\varepsilon \cdot 5^{\alpha-1})5^{\alpha-1} - \left\lceil \frac{b}{12} \right\rceil,$$

where $b = b(\alpha)$ is the smallest integer greater than $4 \cdot 5^{\alpha-2}$ with $b \equiv 5^\alpha \pmod{12}$, $\varepsilon = \varepsilon(\alpha) = 1$ if α is odd, and $\varepsilon = \varepsilon(\alpha) = 2$ if α is even. Then

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \textit{for all } n$$

if and only if

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \textit{for all } n \leq C(\alpha),$$

where λ_α is the least positive reciprocal of 12 modulo 5^α.

Proof. Let

$$f(z) = \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}\eta^b(2 \cdot 5^\alpha z)\left(\frac{\eta^5(z)}{\eta(5z)}\right)^{\varepsilon \cdot 5^{\alpha-1}} = \sum_{n=0}^{\infty} a(n)q^n,$$

where $\eta(z)$ is the Dedekind eta-function, defined by $\eta(z) = q^{1/24}(q;q)_\infty$, $q = e^{2\pi iz}$, $b = b(\alpha)$ is the smallest integer greater than $4\cdot 5^{\alpha-2}$ with $b \equiv 5^\alpha \pmod{12}$, $\varepsilon = \varepsilon(\alpha) = 1$ if α is odd, and $\varepsilon = \varepsilon(\alpha) = 2$ if α is even.

Using results from [4, Theorems 3 and 5] on the properties of $\eta-$products, we find that $f(z)$ is a holomorphic modular form of weight $\frac{b-1}{2} + 2\varepsilon \cdot 5^{\alpha-1}$ and character χ_0, the trivial character, with respect to $\Gamma_0(16 \cdot 5^\alpha)$.

Notice that

$$\left(\frac{\eta^5(z)}{\eta(5z)}\right)^{\varepsilon\cdot 5^{\alpha-1}} = 1 + 5^\alpha \sum_{n=1}^{\infty} h(n)q^n,$$

where the $h(n)$ are integers, and thus the Fourier coefficients of $f(z)$ are congruent to the Fourier coefficients of

$$\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}\eta^b(2\cdot 5^\alpha z)$$

modulo 5^α.

Next, note that in terms of eta-functions,

$$\sum_{n\geq 0} c\phi_2(n)q^n = q^{1/12}\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}.$$

Thus, if we let

$$q^{-\frac{2b\cdot 5^\alpha}{24}}\eta^b(2\cdot 5^\alpha z) = \sum_{n=0}^{\infty} d(2\cdot 5^\alpha n)q^{2\cdot 5^\alpha n},$$

then

$$a\left(5^\alpha n + \lambda_\alpha + \frac{2b\cdot 5^\alpha - 2}{24}\right) \equiv \sum_{m=0}^{\infty} d(2\cdot 5^\alpha m)c\phi_2(5^\alpha n + \lambda_\alpha - 2\cdot 5^\alpha m) \pmod{5^\alpha}.$$

Since $d(0) = 1$, this becomes

$$a\left(5^\alpha n + \lambda_\alpha + \frac{2b\cdot 5^\alpha - 2}{24}\right) \equiv c\phi_2(5^\alpha n + \lambda_\alpha) + \sum_{m=1}^{\infty} d(2\cdot 5^\alpha m)c\phi_2(5^\alpha n + \lambda_\alpha - 2\cdot 5^\alpha m) \pmod{5^\alpha}.$$

By induction, it is easy to see that $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all $n \leq C(\alpha)$ if and only if $a\left(5^\alpha n + \lambda_\alpha + \frac{2b\cdot 5^\alpha - 2}{24}\right) \equiv 0 \pmod{5^\alpha}$ for all $n \leq C(\alpha)$. Hence, we also have that $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all n if and only if $a\left(5^\alpha n + \lambda_\alpha + \frac{2b\cdot 5^\alpha - 2}{24}\right) \equiv 0 \pmod{5^\alpha}$ for all n.

Now notice that $\lambda_\alpha + \frac{2b \cdot 5^\alpha - 2}{24} \equiv 0 \pmod{5^\alpha}$ by hypothesis, so let us consider

$$f_1(z) = f(z)|T_{5^\alpha} = \sum_{n=0}^{\infty} a(5^\alpha n) q^n,$$

which is also a holomorphic modular form of weight $\frac{b-1}{2} + 2\varepsilon \cdot 5^{\alpha-1}$ and character χ_0 with respect to $\Gamma_0(16 \cdot 5^\alpha)$. (See [7, pages 153-175] for a full explanation of the action of the Hecke operators T_p.) We find by Sturm's Theorem that $a(5^\alpha n) \equiv 0 \pmod{5^\alpha}$ for all n if and only if $a(5^\alpha n) \equiv 0 \pmod{5^\alpha}$ for all $n \leq \frac{(\frac{b-1}{2} + 2\varepsilon \cdot 5^{\alpha-1})(16 \cdot 5^\alpha)}{12} \prod_{p|10} \left(1 + \frac{1}{p}\right)$. Therefore $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all n if and only if the congruence holds for all $n \leq C(\alpha)$.

□

For certain values of α, it is not difficult to make modest improvements to Theorem 1. In the case $\alpha = 4$, this modest improvement will bring $C(\alpha)$ more comfortably within the realm of computational feasibility.

Theorem 4. *Let*

$$C(4) := 198745.$$

Then

$$c\phi_2(625n + 573) \equiv 0 \pmod{625} \quad \textit{for all } n$$

if and only if

$$c\phi_2(625n + 573) \equiv 0 \pmod{625} \quad \textit{for all } n \leq C(4).$$

Proof. Let

$$f(z) = \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)} \eta^{44}(625z)\eta^7(1250z)\eta^{10}(2500z) \left(\frac{\eta^5(z)}{\eta(5z)}\right)^{250} = \sum_{n=0}^{\infty} a(n) q^n,$$

where $q = e^{2\pi i z}$. We find that $f(z)$ is a holomorphic modular form of weight 530 and character χ_0, the trivial character, with respect to $\Gamma_0(2500)$.

Notice that

$$\left(\frac{\eta^5(z)}{\eta(5z)}\right)^{250} = 1 + 625 \sum_{n=1}^{\infty} h(n) q^n,$$

where the $h(n)$ are integers, and thus the Fourier coefficients of $f(z)$ are congruent to the Fourier coefficients of

$$\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)} \eta^{44}(625z)\eta^7(1250z)\eta^{10}(2500z)$$

modulo 625.

Recalling that

$$\sum_{n\geq 0} c\phi_2(n)q^n = q^{1/12}\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)},$$

if we let

$$q^{-\frac{61250}{24}}\eta^{44}(625z)\eta^7(1250z)\eta^{10}(2500z) = \sum_{n=0}^{\infty} d(625n)q^{625n},$$

then

$$a\left(625n+573+\frac{61250-2}{24}\right) \equiv \sum_{m=0}^{\infty} d(625m)c\phi_2(625n+573-625m) \pmod{625}.$$

Since $d(0)=1$, this becomes

$$\begin{aligned} a\,(625n+573+2552) &\equiv c\phi_2(625n+573) \\ &\quad + \sum_{m=1}^{\infty} d(625m)c\phi_2(625n+573-625m) \pmod{625}. \end{aligned}$$

By induction, it is easy to see that $c\phi_2(625n+573) \equiv 0 \pmod{625}$ for all $n \leq C(4)$ if and only if $a\,(625n+573+2552) \equiv 0 \pmod{625}$ for all $n \leq C(4)$. Hence, we also have that $c\phi_2(625n+573) \equiv 0 \pmod{625}$ for all n if and only if $a(625n+573+2552) \equiv 0 \pmod{625}$ for all n.

Now notice that $573+2552 \equiv 0 \pmod{625}$, so let us consider

$$f_1(z) = f(z)|T_{625} = \sum_{n=0}^{\infty} a(625n)q^n,$$

which is also a holomorphic modular form of weight 530 and character χ_0 with respect to $\Gamma_0(2500)$. We find by Sturm's Theorem that $a(625n) \equiv 0 \pmod{625}$ for all n if and only if $a(625n) \equiv 0 \pmod{625}$ for all $n \leq \frac{(530)(2500)}{12}\prod_{p|10}\left(1+\frac{1}{p}\right)$. Therefore $c\phi_2(625n+573) \equiv 0 \pmod{625}$ for all n if and only if the congruence holds for all $n \leq C(4)$.

□

3 Calculating the Needed Values of $c\phi_2$

From the above, we can prove the congruences desired for all n after calculating the first M values of $c\phi_2$, for any $M > 5^\alpha C(\alpha) + \lambda_\alpha$. We calculate the necessary terms using recurrences.

The recurrences needed for $c\phi_2(m)$ are easily developed. Recurrences are suitable for calculating the values of $c\phi_2(m)$ for "small" m. This, of course, is the historical approach to the calculation of partition function values. For example, this was the technique used by P. A. MacMahon to compute the first 200 values of $p(m)$ [5, Table IV]. This same table was used by Ramanujan [8] in conjecturing several of the congruences in (5) above.

Theorem 5.

$$\left[\sum_{n\geq 0} c\phi_2(n)q^n\right]\left[\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right] = \left[\sum_{n\geq 0} p(n)q^{2n}\right]\left[\sum_{n\in\mathbb{Z}} q^{n^2}\right]$$

Proof. From Jacobi's Triple Product Identity [1, Theorem 2.8], we see that

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = (q^2;q^2)_\infty (q;q^2)_\infty^2 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}.$$

Also, since

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

it is clear that

$$\sum_{n\geq 0} p(n)q^{2n} = \frac{1}{(q^2;q^2)_\infty}.$$

Then

$$\begin{aligned}
\left[\sum_{n\geq 0} c\phi_2(n)q^n\right]\left[\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right] &= \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4 (q^4;q^4)_\infty} \cdot (q^2;q^2)_\infty (q;q^2)_\infty^2 \\
&= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (q^4;q^4)_\infty^2} \\
&= \frac{1}{(q^2;q^2)_\infty} \cdot \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \\
&= \left[\sum_{n\geq 0} p(n)q^{2n}\right]\left[\sum_{n\in\mathbb{Z}} q^{n^2}\right]
\end{aligned}$$

□

From this theorem, we have the following recurrences:

$$c\phi_2(2k) = p(k) + 2\sum_{m\geq 1}(-1)^{m+1} c\phi_2(2k-m^2) + 2\sum_{m\geq 1} p(k-2m^2)$$

$$c\phi_2(2k+1) = 2\sum_{m\geq 1}(-1)^{m+1} c\phi_2(2k+1-m^2) + 2\sum_{m\geq 0} p(k-2m(m+1))$$

Since $p(n)$ satisfies $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$, where the values in question are the pentagonal numbers, the above recurrences can easily be implemented to calculate several values of $c\phi_2$.

Using these recurrences, we have calculated the necessary 124, 216, 198 values of $c\phi_2$ on a Linux PC with 768MB of RAM and a 600Mhz Pentium III processor. The calculations, all performed modulo 625, were completed in approximately 147 hours of computing time.

With these calculations complete and the congruences checked modulo 625, Theorem 1 has been proven.

4 Closing Remarks

While it would be nice to prove additional cases of (4) using this technique, it is clear that $C(\alpha)$ grows too rapidly to make such an approach feasible. For example, the proof of the $\alpha = 5$ case of (4) would require the calculation of $C(5) = 11279958$ values of $c\phi_2$ in the arithmetic progression $5^5n + \lambda_5$. Hence, we would have to calculate the first 3.5×10^{10} values of $c\phi_2$ (approximately).

Certainly, a proof of Conjecture 1 via modular forms or generating function manipulations is still desired. This was originally requested in [9], and we renew that request here, given the new computational information that is now known about this partition function and the fact that Theorem 1 is proven.

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