Binary Partitions Revisited

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Abstract

The restricted binary partition function $b_k(n)$ enumerates the number of ways to represent n as $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_j}$ with $0 \le a_0 \le a_1 \le \ldots \le a_j < k$. We study the question of how large a power of 2 divides the difference $b_k(2^{r+2}n) - b_{k-2}(2^r n)$ for fixed $k \ge 3$, $r \ge 1$, and all $n \ge 1$.

1 Introduction

Let $b(n)$ denote the number of partitions of the positive integer n into powers of 2. That is, $b(n)$ is the number of ways to represent n as

$$n = 2^{a_0} + 2^{a_1} + \cdots \quad \text{with} \quad a_i \in \mathbb{Z} \quad \text{and} \quad 0 \le a_0 \le a_1 \le \ldots$$

We call $b(n)$ *the binary partition function.*

Churchhouse [2] conjectured that

$$b(2^{r+2}n) - b(2^r n) \equiv 0 \pmod{2^{\lfloor 3r/2 \rfloor + 2}} \quad \text{for} \quad r \ge 1. \tag{1}$$

Moreover, he conjectured that this result is *exact*; i.e. no higher power of 2 divides the left hand side if n is odd. Churchhouse's conjecture was first proven in [6]. Subsequently, others produced proofs, including Gupta [3], [4], [5], and Andrews [1].

In [7] we proved a number of congruences for the restricted m-ary partition function with similar consequences for the ordinary (unrestricted) m-ary partition function. However, in the binary case $m = 2$, these congruences reduce to mere trivialities. The object of this paper is to establish some alternative results for the *restricted* binary partition function $b_k(n)$, which is the number of ways to represent n as

$$n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_j} \quad \text{with} \quad 0 \le a_0 \le a_1 \le \ldots \le a_j < k.$$

We also have that $b_k(n)$ equals the number of representations of n of the form

$$n = c_0 + c_1 2 + c_2 2^2 + \cdots \quad \text{with} \quad 0 \le c_i < 2^k.$$

We now present the two theorems which we prove below.

Theorem 1 *For* $1 \le r \le k-2$ *we have*

$$b_k(2^{r+2}n) - b_{k-2}(2^r n) \equiv 0 \pmod{2^{\lfloor 3r/2 \rfloor + 2}}. \tag{2}$$

Notice that for a given n, $b(n) = b_k(n)$ for sufficiently large k, so that Theorem 1 implies (1).

Although not exact, Theorem 1 is "best possible" in the following sense: For $1 \le r \le k-3$, no higher power of 2 divides the left hand side of (2) if $n \equiv 1 \pmod{2^{k-r-1}}$. Furthermore, for $r = k-2$, we have

$$(3)\quad b_k(2^k n) - b_{k-2}(2^{k-2}n) \equiv 2^{\lfloor 3k/2 \rfloor - 1}\frac{n(n+1)}{2} \pmod{2^{\lfloor 3k/2 \rfloor}}, \quad k \ge 3.$$

If we replace n by $2n$ in (3), we get an exact result, which is the case $t = 1$ of the next theorem.

Theorem 2 *For $k \ge 3$ and $t \ge 1$, we have*

$$b_k(2^{k+t}n) - b_{k-2}(2^{k+t-2}n) \equiv 0 \pmod{2^{\lfloor 3k/2 \rfloor + t - 2}}.$$

Moreover, this result is exact.

We prove Theorems 1 and 2 by considering various aspects of the generating function for $b_k(n)$. Theorem 1 follows from Lemma 1 below, while Theorem 2 follows from Lemma 3. Indeed, Lemmata 1 and 3 give somewhat stronger results than those stated in Theorems 1 and 2, but the stronger results are also more complicated.

2 Auxiliaries

In the following we write $\pi(a)$ for the largest integer π such that 2^π divides the nonzero integer a. Notice that

$$\begin{aligned} \pi(a) < \pi(c) \quad &\text{implies} \quad \pi(\pm a \pm c) = \pi(a), \\ \pi(a) = \pi(c) \quad &\text{implies} \quad \pi(\pm a \pm c) > \pi(a). \end{aligned}$$

We regard $\pi(0) > c$ for any integer c as valid.

All power series in this paper will be elements of $\mathbb{Z}[[q]]$, the ring of formal power series in q with coefficients in $\mathbb{Z}$. We define a $\mathbb{Z}$-linear operator

$$U : \mathbb{Z}[[q]] \longrightarrow \mathbb{Z}[[q]]$$

via

$$U\sum_n a(n)q^n = \sum_n a(2n)q^n.$$

Notice that if $f(q), g(q) \in \mathbb{Z}[[q]]$, then

$$U(f(q)g(q^2)) = (Uf(q))g(q). \tag{4}$$

Moreover, if $f(q) = \sum_n a(n)q^n \in \mathbb{Z}[[q]]$, $g(q) = \sum_n c(n)q^n \in \mathbb{Z}[[q]]$, and M is a positive integer, then we have

$$f(q) \equiv g(q) \pmod{M} \qquad (\text{in } \mathbb{Z}[[q]])$$

if and only if, for all n,

$$a(n) \equiv c(n) \pmod{M} \qquad (\text{in } \mathbb{Z}).$$

In the work below we shall use the following identity for binomial coefficients:

$$\binom{2n+r-1}{r} = \sum_{i=\lceil r/2\rceil}^{r} (-1)^{r-i}2^{2i-r}\binom{i}{r-i}\binom{n+i-1}{i} \tag{5}$$

The truth of this relation follows by expanding both sides of the identity

$$\frac{1}{(1-q)^{2n}} = \frac{1}{(1-q(2-q))^n},$$

and comparing the coefficient of q^r on each side of the equation.

We now begin to develop the machinery needed to prove our two theorems. First, let

$$h_i = h_i(q) = \frac{q}{(1-q)^{i+1}}, \qquad i \geq 0.$$

Then

$$h_i = \sum_{n=1}^{\infty} \binom{n+i-1}{i} q^n, \tag{6}$$

so that

$$Uh_r = \sum_{n=1}^{\infty} \binom{2n+r-1}{r} q^n.$$

It follows from (5) and (6) that

$$Uh_r = \sum_{i=\lceil r/2 \rceil}^{r} (-1)^{r-i} 2^{2i-r} \binom{i}{r-i} h_i \tag{7}$$

for $r \geq 0$.

Next, we recursively define $K_r = K_r(q)$ by

$$K_2 = 2^3 h_2 \quad \text{and} \quad K_{i+1} = U\Big(\frac{1}{1-q} K_i\Big) \tag{8}$$

for $i \geq 2$. We have the following lemma regarding K_r.

Lemma 1 *For $1 \leq i \leq r-1$, there exist $\gamma_r(i) \in \mathbb{Z}$ such that*

$$K_r = \sum_{i=1}^{r-1} \gamma_r(i) h_{i+1}. \tag{9}$$

Moreover,

$$\pi(\gamma_r(i)) \geq \left\lfloor \frac{3r + i^2}{2} \right\rfloor,$$

where equality holds if and only if $i = 1$ or $r + i$ is odd.

Note. In the following we set $\gamma_r(i) = 0$ if $i \geq r$.

Proof. We use induction on r. The lemma is true for $r = 2$ thanks to (8). Suppose that the lemma is true for r replaced by $r-1$ for some $r \geq 3$. Then we have

$$K_{r-1} = \sum_{i=1}^{r-2} \gamma_{r-1}(i) h_{i+1}, \tag{10}$$

and

$$\pi(\gamma_{r-1}(i)) \geq \left\lfloor \frac{3(r-1) + i^2}{2} \right\rfloor, \tag{11}$$

where equality holds if and only if $i = 1$ or $r + i$ is even (and $1 \le i \le r - 2$). By (10), (8), and (7), we find

$$\begin{aligned} K_r &= \sum_{j=1}^{r-2} \gamma_{r-1}(j) U h_{j+2} \\ &= \sum_{j=1}^{r-2} \gamma_{r-1}(j) \sum_{i=\lceil j/2 \rceil + 1}^{j+2} (-1)^{i+j} 2^{2i-j-2} \binom{i}{j+2-i} h_i \\ &= \sum_{i=2}^{r} \sum_{j=\max(1,i-2)}^{\min(r-2,2i-2)} (-1)^{i+j} 2^{2i-j-2} \binom{i}{j+2-i} \gamma_{r-1}(j) h_i \\ &= \sum_{i=1}^{r-1} \sum_{j=\max(1,i-1)}^{\min(r-2,2i)} (-1)^{i+j+1} 2^{2i-j} \binom{i+1}{j+1-i} \gamma_{r-1}(j) h_{i+1}. \end{aligned}$$

Thus (9) holds with

$$\gamma_r(i) = \sum_{j=\max(1,i-1)}^{\min(r-2,2i)} (-1)^{i+j+1} 2^{2i-j} \binom{i+1}{j+1-i} \gamma_{r-1}(j), \tag{12}$$

so that all values $\gamma_r(i)$ are integers. Now we have

$$\gamma_r(1) = -2^2 \gamma_{r-1}(1) + \gamma_{r-1}(2),$$

where

$$\pi(2^2 \gamma_{r-1}(1)) = 2 + \left\lfloor \frac{3(r-1)+1}{2} \right\rfloor = \left\lfloor \frac{3r+2}{2} \right\rfloor.$$

If r is odd, then

$$\pi(2^2 \gamma_{r-1}(1)) = \left\lfloor \frac{3r+1}{2} \right\rfloor,$$

while

$$\pi(\gamma_{r-1}(2)) > \left\lfloor \frac{3(r-1)+2^2}{2} \right\rfloor = \left\lfloor \frac{3r+1}{2} \right\rfloor,$$

so that

$$\pi(\gamma_r(1)) = \left\lfloor \frac{3r+1}{2} \right\rfloor. \tag{13}$$

If r is even, then

$$\pi(2^2 \gamma_{r-1}(1)) = \left\lfloor \frac{3r+1}{2} \right\rfloor + 1,$$

while

$$\pi(\gamma_{r-1}(2)) = \left\lfloor \frac{3(r-1)+2^2}{2} \right\rfloor = \left\lfloor \frac{3r+1}{2} \right\rfloor,$$

and (13) holds in this case also.

Next, let $2 \le i \le r-1$. By (12), we then have

$$\gamma_r(i) = 2^{i+1}\gamma_{r-1}(i-1) - 2^i(i+1)\gamma_{r-1}(i) + \Delta_1, \tag{14}$$

where, by (11),

$$\pi(\Delta_1) \ge \min_{j \ge i+1}(2i - j + \left\lfloor \frac{3(r-1)+j^2}{2} \right\rfloor) \ge \left\lfloor \frac{3r+i^2}{2} \right\rfloor + 2. \tag{15}$$

Now consider $i = 2$. We have

$$\pi(2^3\gamma_{r-1}(1)) = 3 + \left\lfloor \frac{3(r-1)+1}{2} \right\rfloor = \left\lfloor \frac{3r+2^2}{2} \right\rfloor.$$

If r is odd, then

$$\pi(2^2 \cdot 3\gamma_{r-1}(2)) > 2 + \left\lfloor \frac{3(r-1)+2^2}{2} \right\rfloor > \left\lfloor \frac{3r+2^2}{2} \right\rfloor,$$

so that, by (14) and (15),

$$\pi(\gamma_r(2)) = \left\lfloor \frac{3r+2^2}{2} \right\rfloor.$$

If r is even, then

$$\pi(2^2 \cdot 3\gamma_{r-1}(2)) = 2 + \left\lfloor \frac{3(r-1)+2^2}{2} \right\rfloor = \left\lfloor \frac{3r+2^2}{2} \right\rfloor,$$

and it follows that

$$\pi(\gamma_r(2)) > \left\lfloor \frac{3r+2^2}{2} \right\rfloor.$$

Finally, if $3 \le i \le r-1$, then

$$\pi(2^i(i+1)\gamma_{r-1}(i)) \ge i + \left\lfloor \frac{3(r-1)+i^2}{2} \right\rfloor > \left\lfloor \frac{3r+i^2}{2} \right\rfloor,$$

so that, by (11), (14), and (15),

$$\pi(\gamma_r(i)) \ge \left\lfloor \frac{3r+i^2}{2} \right\rfloor$$

with equality if and only if $r+i$ is odd. This implies the result stated in Lemma 1. ■

3 Proof of Theorem 1

With $b_k(0) = 1$, the generating function for $b_k(n)$ is

$$B_k(q) = \sum_{n=0}^{\infty} b_k(n) q^n = \prod_{i=0}^{k-1} \frac{1}{1-q^{2^i}}, \qquad k \geq 0,$$

where, in particular, $B_0(q) = 1$. Notice that, for $k \geq 1$,

$$B_k(q) = \frac{1}{1-q} B_{k-1}(q^2). \tag{16}$$

Thanks to (4), we have for $k \geq 2$,

$$\begin{aligned} UB_k(q) &= (U\frac{1}{1-q})B_{k-1}(q) \\ &= \frac{1}{1-q}B_{k-1}(q) \\ &= \frac{1}{(1-q)^2}B_{k-2}(q^2) \quad \text{from (16)} \\ &= \sum_{n=0}^{\infty}(n+1)q^n B_{k-2}(q^2). \end{aligned}$$

Furthermore,

$$U^2B_k(q) = \sum_{n=0}^{\infty}(2n+1)q^n B_{k-2}(q),$$

so that, for $k \geq 3$,

$$\begin{aligned} U^2B_k(q) - B_{k-2}(q) &= \sum_{n=1}^{\infty}(2n+1)q^n B_{k-2}(q) \\ &= (2h_1 + h_0)B_{k-2}(q) \\ &= (2h_2 + h_1)B_{k-3}(q^2). \end{aligned}$$

By (7), we now have

$$\begin{aligned} U^3B_k(q) - UB_{k-2}(q) &= (2Uh_2 + Uh_1)B_{k-3}(q) \\ &= 2^3 h_2 B_{k-3}(q) \\ &= K_2 B_{k-3}(q). \end{aligned}$$

Moreover, since

$$U(K_iB_{k-i-1}(q)) = U(\frac{1}{1-q}K_iB_{k-i-2}(q^2)) = K_{i+1}B_{k-i-2}(q),$$

induction on r gives

$$U^{r+2}B_k(q) - U^rB_{k-2}(q) = K_{r+1}B_{k-r-2}(q)$$

for $1 \le r \le k-2$. Thus we have

$$\sum_{n=1}^{\infty}(b_k(2^{r+2}n) - b_{k-2}(2^rn))q^n = K_{r+1}B_{k-r-2}(q) \tag{17}$$

for $1 \le r \le k-2$. Theorem 1 now follows from Lemma 1.

Next we turn to the remarks following the statement of Theorem 1. For $r \ge 1$, we have, by Lemma 1,

$$K_{r+1}(q) \equiv 2^{\lfloor 3r/2 \rfloor+2}h_2(q) \pmod{2^{\lfloor 3r/2 \rfloor+3}}. \tag{18}$$

If we now put $r = k-2$ in (17), (3) follows by (18) and (6).

Let

$$\sum_{n=1}^{\infty} d_r(n)q^n = h_2(q)B_{k-r-2}(q). \tag{19}$$

Since

$$B_k(q) \equiv \prod_{i=0}^{k-1}\frac{1}{(1-q)^{2^i}} \equiv \frac{1}{(1-q)^{2^k-1}} \pmod 2,$$

we then have, for $1 \le r \le k-3$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_r(n+1)q^n &\equiv \frac{1}{(1-q)^{2^{k-r-2}+2}} \\ &\equiv \frac{1}{(1-q^2)^{2^{k-r-3}+1}} \pmod 2,\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty} d_r(2n+1)q^n &\equiv \frac{1}{(1-q)^{2^{k-r-3}+1}} \\ &\equiv \frac{1}{1-q}\cdot\frac{1}{1-q^{2^{k-r-3}}} \pmod 2.\end{aligned}$$

Repeated application of (4) now gives

$$\sum_{n=0}^{\infty} d_r(2^{k-r-2}n+1)q^n \equiv \frac{1}{(1-q)^2} \equiv \frac{1}{1-q^2} \pmod 2,$$

so that

$$d_r(2^{k-r-1}n+1) \equiv 1 \pmod 2, \tag{20}$$

for $1 \leq r \leq k-3$. From (17)–(20) it now follows that, for $1 \leq r \leq k-3$, the left hand side of (2) is not divisible by $2^{\lfloor 3r/2 \rfloor+3}$ if $n \equiv 1 \pmod{2^{k-r-1}}$.

4 Proof of Theorem 2

By putting $r = k-2$ in (17), we see that

$$\sum_{n=1}^{\infty} (b_k(2^{k+t}n) - b_{k-2}(2^{k+t-2}n)q^n = U^t K_{k-1}(q). \tag{21}$$

With the goal of proving Theorem 2, we prove the following two lemmas regarding $U^t K_r(q)$. We first consider the $t = 1$ case, $UK_r(q)$, as a basis case.

Lemma 2 *For $r \geq 2$, there exist $\delta_{r,1}(i) \in \mathbb{Z}$ such that*

$$UK_r = \sum_{i=1}^{r} \delta_{r,1}(i) h_i, \tag{22}$$

where

$$\pi(\delta_{r,1}(1)) = \left\lfloor \frac{3r+1}{2} \right\rfloor, \tag{23}$$

and

$$\pi(\delta_{r,1}(i)) \geq \left\lfloor \frac{3r+i^2+1}{2} \right\rfloor \quad \text{for } i = 2, \ldots, r. \tag{24}$$

Moreover, (24) *holds with equality if and only if $i = 2$ or $r + i$ is even.*

Proof. By (9) and (7), we have

$$
\begin{aligned}
UK_r &= \sum_{j=1}^{r-1} \gamma_r(j) U h_{j+1} \\
&= \sum_{j=1}^{r-1} \gamma_r(j) \sum_{i=\lceil \frac{j+1}{2} \rceil}^{j+1} (-1)^{j+1-i} 2^{2i-j-1} \binom{i}{j+1-i} h_i \\
&= \sum_{i=1}^{r} \sum_{j=\max(1,i-1)}^{\min(r-1,2i-1)} (-1)^{i+j+1} 2^{2i-j-1} \binom{i}{j+1-i} \gamma_r(j) h_i.
\end{aligned}
$$

Thus (22) holds with

$$
\delta_{r,1}(i) = \sum_{j=\max(1,i-1)}^{\min(r-1,2i-1)} (-1)^{i+j+1} 2^{2i-j-1} \binom{i}{j+1-i} \gamma_r(j),
$$

and all values $\delta_{r,1}(i)$ are integers. Moreover, $\delta_{r,1}(1) = -\gamma_r(1)$, so by Lemma 1, (23) holds.

For $2 \le i \le r$, we have

$$
\delta_{r,1}(i) = 2^i \gamma_r(i-1) - 2^{i-1} i \gamma_r(i) + \Delta_2, \tag{25}
$$

where

$$
\pi(\Delta_2) \ge \min_{j \ge i+1} (2i - j - 1 + \left\lfloor \frac{3r + j^2}{2} \right\rfloor) \ge \left\lfloor \frac{3r + i^2 + 1}{2} \right\rfloor + 2. \tag{26}
$$

Note that

$$
\pi(2^i \gamma_r(i-1)) \ge i + \left\lfloor \frac{3r + (i-1)^2}{2} \right\rfloor = \left\lfloor \frac{3r + i^2 + 1}{2} \right\rfloor \tag{27}
$$

with equality if and only if $i = 2$ or $r + i$ is even. Furthermore,

$$
\pi(2^{i-1} i \gamma_r(i)) \ge i - 1 + \pi(i) + \left\lfloor \frac{3r + i^2}{2} \right\rfloor > \left\lfloor \frac{3r + i^2 + 1}{2} \right\rfloor, \tag{28}
$$

where we look separately at the cases $i = 2$ and $i \ge 3$. Combining (25)–(28) completes the proof of Lemma 2. ∎

Lemma 3 *For $r \geq 2$ and $t \geq 1$, there exist $\delta_{r,t}(i) \in \mathbb{Z}$ such that*

$$U^t K_r = \sum_{i=1}^{r} \delta_{r,t}(i) h_i, \tag{29}$$

where

$$\pi(\delta_{r,t}(1)) = \left\lfloor \frac{3r+1}{2} \right\rfloor + t - 1, \tag{30}$$

and

$$\pi(\delta_{r,t}(i)) \geq \left\lfloor \frac{3r+i^2+1}{2} \right\rfloor + i(t-1) \quad \text{for } i = 2, \ldots, r. \tag{31}$$

Moreover, (31) *holds with equality if and only if $i = 2$ or $r + i$ is even.*

Proof. We use induction on t. By Lemma 2, Lemma 3 is true for $t = 1$. Next, suppose that Lemma 3 is true for t replaced by $t-1$ for some $t \geq 2$.

Using (7), we get

$$\begin{aligned} U^t K_r &= \sum_{j=1}^{r} \delta_{r,t-1}(j) U h_j \\ &= \sum_{j=1}^{r} \delta_{r,t-1}(j) \sum_{i=\lceil j/2 \rceil}^{j} (-1)^{j-i} 2^{2i-j} \binom{i}{j-i} h_i \\ &= \sum_{i=1}^{r} \sum_{j=i}^{\min(r,2i)} (-1)^{i+j} 2^{2i-j} \binom{i}{j-i} \delta_{r,t-1}(j) h_i, \end{aligned}$$

so that (29) holds with

$$\delta_{r,t}(i) = \sum_{j=i}^{\min(r,2i)} (-1)^{i+j} 2^{2i-j} \binom{i}{j-i} \delta_{r,t-1}(j),$$

and all values $\delta_{r,t}(i)$ are integers.

Now we have

$$\delta_{r,t}(1) = 2\delta_{r,t-1}(1) - \delta_{r,t-1}(2).$$

By the induction assumption, we have

$$\pi(2\delta_{r,t-1}(1)) = 1 + \left\lfloor \frac{3r+1}{2} \right\rfloor + t - 2,$$

and

$$\pi(\delta_{r,t-1}(2)) = \left\lfloor \frac{3r+2^2+1}{2} \right\rfloor + 2(t-2) > \left\lfloor \frac{3r+1}{2} \right\rfloor + t - 1.$$

Thus, (30) follows.

For $2 \le i \le r$, we have

$$\delta_{r,t}(i) = 2^i \delta_{r,t-1}(i) + \Delta_3, \tag{32}$$

where

$$\pi(\Delta_3) \ge \min_{j \ge i+1} (2i - j + \left\lfloor \frac{3r+j^2+1}{2} \right\rfloor + j(t-2)),$$

so that

$$\pi(\Delta_3) > \left\lfloor \frac{3r+i^2+1}{2} \right\rfloor + i(t-1). \tag{33}$$

Moreover,

$$\pi(2^i \delta_{r,t-1}(i)) \ge \left\lfloor \frac{3r+i^2+1}{2} \right\rfloor + i(t-1), \tag{34}$$

with equality if and only if $i = 2$ or $r + i$ is even. Combining (32)–(34) completes the proof of Lemma 3. ■

We are now in a position to prove Theorem 2. For $k \ge 3$ and $t \ge 1$, we have by Lemma 3 and (6),

$$U^t K_{k-1} \equiv \delta_{k-1,t}(1) h_1 \equiv \delta_{k-1,t}(1) \sum_{n=1}^{\infty} n q^n \pmod{2^{\lfloor 3k/2 \rfloor + 2t - 1}}.$$

In particular, by (30) and (21),

$$b_k(2^{k+t} n) - b_{k-2}(2^{k+t-2} n) \equiv 2^{\lfloor 3k/2 \rfloor + t - 2} n \pmod{2^{\lfloor 3k/2 \rfloor + t - 1}},$$

and the proof of Theorem 2 is complete.

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