Parity Results for p-Regular Partitions with Distinct Parts

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Abstract

We consider the partition function $b'_p(n)$, which counts the number of partitions of the integer n into distinct parts with no part divisible by the prime p. We prove the following: Let p be a prime greater than 3 and let r be an integer between 1 and $p-1$, inclusively, such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all nonnegative integers n, $b'_p(pn+r) \equiv 0 \pmod 2$.

1 Introduction

A **partition** λ of the nonnegative integer n is a nonincreasing sequence of nonnegative integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_r = n$. Each value λ_i, $1 \leq i \leq r$, is called a **part** of the partition. The number of partitions of n is counted by the **partition function** $p(n)$.

A partition $\lambda_1, \lambda_2, \ldots, \lambda_r$ of n is **p-regular** if no part λ_i, $1 \leq i \leq r$, is divisible by p. The function which enumerates the p-regular partitions of n is often denoted $b_p(n)$. These functions have been the focus of much study in recent years [1], [4], [5]. The function $b_p(n)$ is of particular interest for prime p, as it yields the number of irreducible p-modular representations of the symmetric group S_n [7].

The function which counts those p-regular partitions of n which consist of distinct parts will be denoted $b'_p(n)$ in this note. Such functions have appeared in a variety of recent works. For example, parity results for $b'_2(n)$, the number of partitions of n into distinct odd parts, were found by Hirschhorn [6]. Moreover, the function $b'_5(n)$ was studied by Andrews, Bessenrodt, and Olsson [3] as it relates to representation theory.

2 Main Results

Our main goal here is to prove the following parity result for b'_p by elementary means:

Theorem 2.1. *Let p be a prime greater than 3 and let r be an integer between 1 and $p-1$, inclusively, such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all nonnegative integers n, $b'_p(pn+r) \equiv 0 \pmod 2$.*

Before proving Theorem 2.1, we mention two propositions. The proofs of these can be found in [2, Chapter 1].

Proposition 2.2. *The generating function for $p(n)$ is given by*

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}$$

where $(a;b)_\infty = (1-a)(1-ab)(1-ab^2)(1-ab^3)\ldots$

Proposition 2.3 (Euler's Pentagonal Number Theorem).

$$(q;q)_\infty = \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{3}{2}m^2-\frac{1}{2}m}$$

With these two tools in hand, we turn to the proof of Theorem 2.1.

Proof. Note that the generating function for $b'_p(n)$ is given by

$$\sum_{n\geq 0} b'_p(n)q^n = \frac{(-q;q)_\infty}{(-q^p;q^p)_\infty}.$$

Then we see that

$$\begin{aligned}
\sum_{n\geq 0} b'_p(n)q^n &= (-q;q)_\infty \frac{(q^p;q^p)_\infty}{(q^{2p};q^{2p})_\infty} \\
&\equiv (q;q)_\infty \frac{(q^p;q^p)_\infty}{(q^p;q^p)^2_\infty} \pmod 2 \quad \text{since } 1-q \equiv 1+q \pmod 2 \\
&= (q;q)_\infty \frac{1}{(q^p;q^p)_\infty} \\
&= (q;q)_\infty \sum_{k\geq 0} p(k)q^{pk}
\end{aligned}$$

thanks to Proposition 2.2. But this implies

$$\sum_{n\geq 0} b'_p(n)q^n \equiv \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}m^2-\frac{1}{2}m} \sum_{k\geq 0} p(k)q^{pk} \pmod 2 \tag{1}$$

by Proposition 2.3.

Now we assume

$$pn + r = \frac{3}{2}m^2 - \frac{1}{2}m + pk$$

for some integers m and k. Then we know

$$r \equiv \frac{3}{2}m^2 - \frac{1}{2}m \pmod p.$$

Hence,

$$\begin{aligned}
24r + 1 &\equiv 36m^2 - 12m + 1 \pmod p \\
&\equiv (6m-1)^2 \pmod p.
\end{aligned}$$

But this contradicts the assumption that $24r+1$ is a quadratic nonresidue modulo p. Therefore, $pn+r$ can never be represented as $\frac{3}{2}m^2 - \frac{1}{2}m + pk$ for integers m and k. Thus, by (1), we know

$$b'_p(pn+r) \equiv 0 \pmod 2.$$

□

We note that, for each prime $p > 3$, Theorem 2.1 guarantees $\frac{p-1}{2}$ different congruences modulo 2 for the function b'_p, which is a very satisfying result.

References

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