Beyond Mere Convergence

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Abstract

In this article, I suggest that calculus instruction should include a wider variety of examples of convergent and divergent series than is usually demonstrated. In particular, a number of convergent series, such as $\sum_{k\geq 1} \frac{k^3}{2^k}$, are considered, and their exact values are found in a straightforward manner. We explore and utilize a number of mathematical topics, including manipulation of certain power series and recurrences.

During my most recent spring break, I read William Dunham's book *Euler: The Master of Us All* [3]. I was thoroughly intrigued by the material presented and am certainly glad I selected it as part of the week's reading.

Of special interest were Dunham's comments on series manipulations and the power series identities developed by Euler and his contemporaries, for I had just completed teaching convergence and divergence of infinite series in my calculus class. In particular, Dunham [3, p. 47-48] presents Euler's proof of the Basel Problem, a challenge from Jakob Bernoulli to determine the exact value of the sum $\sum\limits_{k\geq 1} \frac{1}{k^2}$. Euler was the first to solve this problem by proving that the sum equals $\frac{\pi^2}{6}$.

I was reminded of my students' interest in this result when I shared it with them just weeks before. I had already mentioned to them that exact values for relatively few families of convergent series could be determined. The obvious examples are geometric series $\sum\limits_{k\geq 0} r^k$ (with $|r| < 1$) and telescoping series. I also remembered their disappointment when I observed that the exact numerical value of most convergent series cannot be determined in a straightforward way. I tried to excite them with the notion that the convergence or divergence of a given series could be determined via the Integral Test, Limit Comparison Test, Ratio or Root Test, but this was received with little enthusiasm.

But now I return to Dunham's book. In [3, p. 41], Dunham notes that Jakob Bernoulli [2, p. 248-249] proved

$$\sum_{k\geq 1} \frac{k^2}{2^k} = 6 \tag{1}$$

and

$$\sum_{k\geq 1} \frac{k^3}{2^k} = 26. \tag{2}$$

Many teachers of calculus will recognize at least two things about (1) and (2). First, these series are made-to-order examples to demonstrate convergence with the Ratio Test. Such examples, where the summands are defined by the ratio of a polynomial and an exponential function, can be found in a number of calculus texts, such as [4] and [5]. Second - a much more negative admission - is that we rarely teach students how to prove equalities like (1) and (2). We usually stop at demonstrating that such series converge, and move on to other matters. This is the case with the two calculus texts mentioned above, and it is an unfortunate situation to say the least.

I contend that students of first-year calculus would be better served if we provided a few more tools to them for finding **exact** values of convergent infinite series. Oddly enough, the series in (1) and (2) are ideal for such a task.

My goal in this note is to present two approaches to finding the exact value of

$$a(m,n) := \sum_{k\geq 1} \frac{k^n}{m^k}$$

with $|m| > 1$ and $n \in \mathbb{N} \cup \{0\}$ (of which Bernoulli's examples (1) and (2) are special cases).

We begin by noting that, for each $|m| > 1$, $\left|\frac{1}{m}\right| < 1$, so that $a(m,0)$ is a convergent geometric series. Moreover,

$$\begin{aligned}
a(m,0) &= \sum_{k\geq 1} \frac{1}{m^k} \\
&= \frac{1}{m} + \sum_{k\geq 2} \left(\frac{1}{m}\right)^k \\
&= \frac{1}{m} + \frac{1}{m} \sum_{k\geq 1} \left(\frac{1}{m}\right)^k \\
&= \frac{1}{m} + \frac{1}{m} a(m,0).
\end{aligned}$$

Solving for $a(m,0)$, we see that it equals $\frac{1}{m-1}$. Of course, this result easily follows from the usual formula for the sum of a convergent geometric series.

Next, we obtain a recurrence for $a(m,n)$, $n \geq 1$, in terms of $a(m,j)$ for $j < n$. Note that

$$\begin{aligned}
a(m,n) &= \sum_{k\geq 1} \frac{k^n}{m^k} \\
&= \frac{1}{m} + \sum_{k\geq 2} \frac{k^n}{m^k} \\
&= \frac{1}{m} + \frac{1}{m} \sum_{k\geq 1} \frac{(k+1)^n}{m^k} \\
&= \frac{1}{m} \left[1 + \sum_{k\geq 1} \frac{(k+1)^n}{m^k} \right].
\end{aligned}$$

The argument up to this point is exactly that used in finding the formula for $a(m,0)$ above. We now employ the binomial theorem, a tool that should be in the repertoire of first-year calculus students.

$$\begin{aligned}
a(m,n) &= \frac{1}{m}\left[1+\sum_{k\geq 1}\frac{\left(\sum_{j=0}^{n}\binom{n}{j}k^j\right)}{m^k}\right] \\
&= \frac{1}{m}\left[1+\sum_{j=0}^{n}\binom{n}{j}\sum_{k\geq 1}\frac{k^j}{m^k}\right] \\
&= \frac{1}{m}\left[1+\sum_{j=0}^{n-1}\binom{n}{j}\sum_{k\geq 1}\frac{k^j}{m^k}+\sum_{k\geq 1}\frac{k^n}{m^k}\right] \\
&= \frac{1}{m}\left[1+\sum_{j=0}^{n-1}\binom{n}{j}a(m,j)+a(m,n)\right] \\
&= \frac{1}{m}a(m,n)+\frac{1}{m}\left[1+\sum_{j=0}^{n-1}\binom{n}{j}a(m,j)\right]
\end{aligned}$$

Solving for $a(m,n)$ yields

$$\left(1-\frac{1}{m}\right)a(m,n)=\frac{1}{m}\left[1+\sum_{j=0}^{n-1}\binom{n}{j}a(m,j)\right]$$

or

$$a(m,n)=\left(\frac{1}{m-1}\right)\left[1+\sum_{j=0}^{n-1}\binom{n}{j}a(m,j)\right]. \tag{3}$$

As a sidenote, it is interesting to see from (3) that, for rational values of m, the numerical value of $a(m,n)$ must be rational for all $n\geq 0$. This can be proven via induction on n. We noted above that $a(m,0)=\frac{1}{m-1}$ which is rational as long as m is rational. Then, assuming $a(m,j)$ is rational for $0\leq j\leq n-1$, (3) implies $a(m,n)$ is also rational. Hence, no values such as $\frac{\pi^2}{6}$ will arise as values for $a(m,n)$ whenever m is rational.

The recurrence in (3) can be used to calculate with relative ease the **exact** value of

$$a(m,n)=\sum_{k\geq 1}\frac{k^n}{m^k}$$

for all $|m| > 1$ and $n \in \mathbb{N} \cup \{0\}$. For example, since

$$a(2,0) = \sum_{k \geq 1} \frac{1}{2^k} = 1,$$

we have

$$\begin{aligned} a(2,1) &= \sum_{k\geq 1} \frac{k}{2^k} \\ &= \left(\frac{1}{2-1}\right)\left[1 + \binom{1}{0} a(2,0)\right] \\ &= 1 + 1 = 2, \end{aligned}$$

and

$$\begin{aligned} a(2,2) &= \sum_{k\geq 1} \frac{k^2}{2^k} \\ &= 1 + \binom{2}{0} a(2,0) + \binom{2}{1} a(2,1) \\ &= 1 + 1 + 2 \cdot 2 = 6, \end{aligned}$$

which is the result labeled (1). Finally,

$$\begin{aligned} a(2,3) &= \sum_{k\geq 1} \frac{k^3}{2^k} \\ &= 1 + \binom{3}{0} a(2,0) + \binom{3}{1} a(2,1) + \binom{3}{2} a(2,2) \\ &= 1 + 1 + 3 \cdot 2 + 3 \cdot 6 = 26, \end{aligned}$$

which is (2).

Of course, recurrence (3) could be used to calculate $a(m,n)$ for larger values of m and n. However, this might prove tedious for extremely large values of n. With this in mind, we now approach the calculation of $a(m,n)$ from a second point of view.

We begin with the familiar power series representation for the function $\dfrac{1}{1-x}$:

(4) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots, \text{ where } |x| < 1$$

Andrews [1] recently extolled the virtues of (4) in the study of calculus. Our goal in this section is to manipulate (4) via differentiation and multiplication to obtain a new power series of the form

$$f_n(x) := x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \ldots = \sum_{k \geq 1} k^n x^k$$

for a fixed positive integer n. This is done by applying the $x\frac{d}{dx}$ operator to $\frac{1}{1-x}$ n times. Then $a(m, n)$ equals $f_n\left(\dfrac{1}{m}\right)$, which is easily computed once $f_n(x)$ is written as a rational function. (Note that we define $f_0(x)$ by $f_0(x) := x\left(\frac{1}{1-x}\right) = \sum_{k\geq 1} x^k$.)

As an example, we apply the $x\frac{d}{dx}$ operator to $\frac{1}{1-x}$ and get

$$x\frac{d}{dx}\left(\frac{1}{1-x}\right) = x\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \ldots)$$

or

$$f_1(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots = \sum_{k\geq 1} kx^k.$$

Hence,

$$\sum_{k\geq 1} \frac{k}{2^k} = f_1\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

We can apply the $x\frac{d}{dx}$ operator to $\frac{1}{1-x}$ twice to obtain $f_2(x)$:

$$\begin{aligned} f_2(x) &= x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) \\ &= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\ &= \frac{x^2+x}{(1-x)^3}. \end{aligned}$$

Thus,

$$f_2(x) = \frac{x^2+x}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \ldots = \sum_{k\geq 1} k^2x^k.$$

Hence,

$$\sum_{k\geq 1}\frac{k^2}{2^k} = f_2\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = 6$$

upon simplification. This, as we have already seen, is (1).

Additional applications of the $x\frac{d}{dx}$ operator can be performed to yield

$$\begin{aligned}
f_1(x) &= \frac{x}{(1-x)^2} = \sum_{k\geq 1} kx^k, \\
f_2(x) &= \frac{x^2+x}{(1-x)^3} = \sum_{k\geq 1} k^2x^k, \\
f_3(x) &= \frac{x^3+4x^2+x}{(1-x)^4} = \sum_{k\geq 1} k^3x^k, \\
f_4(x) &= \frac{x^4+11x^3+11x^2+x}{(1-x)^5} = \sum_{k\geq 1} k^4x^k, \\
f_5(x) &= \frac{x^5+26x^4+66x^3+26x^2+x}{(1-x)^6} = \sum_{k\geq 1} k^5x^k, \text{ and} \\
f_6(x) &= \frac{x^6+57x^5+302x^4+302x^3+57x^2+x}{(1-x)^7} = \sum_{k\geq 1} k^6x^k.
\end{aligned}$$

We see that

$$f_n(x) = \frac{g_n(x)}{(1-x)^{n+1}}$$

for each $n \geq 1$ where $g_n(x)$ is a certain polynomial of degree n. Indeed, the functions $g_n(x)$ are well-known. Upon searching N.J.A. Sloane's On-Line Encyclopedia of Integer Sequences [6] for the sequence

$$1, 1, 1, 1, 4, 1, 1, 11, 11, 1, 1, 26, 66, 26, 1, \ldots,$$

which is the sequence of coefficients of the polynomials $g_n(x)$, we discover that these are the **Eulerian numbers** $e(n, j)$. They are defined, for each value of j and n satisfying $1 \leq j \leq n$, by

(5) $\quad e(n, j) = je(n-1, j) + (n-j+1)e(n-1, j-1)$ with $e(1, 1) = 1$.

With this notation, it appears that, for $n \geq 1$,

$$f_n(x) = \frac{\sum_{j=1}^{n} e(n,j)x^j}{(1-x)^{n+1}}.$$

Using (5), this assertion can be proven in a straightforward manner via induction. Moreover, we know from [6, Sequence A008292] that

$$e(n,j) = \sum_{\ell=0}^{j} (-1)^\ell (j-\ell)^n \binom{n+1}{\ell}.$$

This can be used to write the rational version of $f_n(x)$ for any $n \geq 1$ in a timely way. So, for example, we see that

$$f_8(x) = \frac{x^8 + 247x^7 + 4293x^6 + 15619x^5 + 15619x^4 + 4293x^3 + 247x^2 + x}{(1-x)^9},$$

which implies

$$\sum_{k\geq 1} \frac{k^8}{5^k} = f_8\left(\frac{1}{5}\right) = \frac{1139685}{2048}.$$

We have thus seen two different ways to compute the exact value of $\sum_{k\geq 1} \frac{k^n}{m^k}$ with $|m| > 1$ and $n \in \mathbb{N} \cup \{0\}$, one with a recurrence and one with power series. I encourage us all to share at least one of these techniques with our students the next time we are exploring infinite series.

References

[1] G. Andrews, *The Geometric Series in Calculus*, American Mathematical Monthly **105**, no. 1 (1998), 36-40.

[2] J. Bernoulli, *Tractatus de seriebus infinitis*, 1689.

[3] W. Dunham, *Euler: The Master of Us All*, The Dolciani Mathematical Expositions, no. 22, Mathematical Association of America, Washington, D.C., 1999.

[4] C. Edwards and D. Penney, *Calculus with Analytic Geometry*, Fifth Edition, Prentice Hall, 1998.

[5] R. Larson, R. Hostetler, and B. Edwards, *Calculus: Early Transcendental Functions*, Second Edition, Houghton Mifflin Company, 1999.

[6] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/∼njas/sequences/.

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Prior to going to Cedarville he received his Ph.D. in mathematics in 1992 from Penn State, where he met his wife Mary. James truly enjoys spending time with Mary and their five children. He agrees with Euler that mathematics can often be enjoyed and discovered with a child in his arms or playing round his feet.