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NEW CONGRUENCES FOR GENERALIZED FROBENIUS PARTITIONS WITH 2 OR 3 COLORS

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Abstract. The goal of this paper is to prove new congruences involving 2–colored and 3–colored generalized Frobenius partitions of n which extend the work of George Andrews and Louis Kolitsch.

Section 1. Introduction.

Generalized Frobenius partitions have been the focus of study for many mathematicians in the last few years. In 1984, George Andrews [1] introduced these objects, known simply as F–partitions, and two partition functions related to them. In particular, he introduced $c\phi_m(n)$, the number of F–partitions of n with m colors, and proved that

$$c\phi_m(n) \equiv 0 \pmod{m^2} \tag{1}$$

if m is prime and m does not divide n.

In an effort to extend Andrews' work, Louis Kolitsch [2, 3] introduced a new partition function, $\overline{c\phi_m}(n)$, which denotes the number of F–partitions of n with m colors whose order is m under cyclic permutation of the m colors. Kolitsch went on to prove a result involving $\overline{c\phi_m}(n)$ which is quite similar to (1) above. Namely, he proved that, for all $n \geq 1$ and for any $m \geq 2$,

$$\overline{c\phi_m}(n) \equiv 0 \pmod{m^2}. \tag{2}$$

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Based on the work of Kolitsch [3], this author has been able to slightly extend (2) above. In a short note [5], the following congruences were proven to hold for all $n \geq 1$:

$$\begin{aligned} \overline{c\phi_5}(5n) &\equiv 0 \pmod{5^3}, \\ \overline{c\phi_7}(7n) &\equiv 0 \pmod{7^3}, \quad \text{and} \\ \overline{c\phi_{11}}(11n) &\equiv 0 \pmod{11^3}. \end{aligned} \tag{3}$$

The goal of this paper is to prove two new congruences similar to (3). Specifically, we will prove the following:

Theorem 1. *For all $n \geq 1$,*

$$\overline{c\phi_2}\,(2n) \equiv 0 \pmod{2^3}, \quad \textit{and} \tag{4}$$

$$\overline{c\phi_3}\,(3n) \equiv 0 \pmod{3^4}. \tag{5}$$

The proof of Theorem 1 will only involve elementary techniques, relying on two well–known results of Jacobi. The first of the two results is Jacobi's Triple Product Identity, which states that

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} \left(1 - q^{2n+2}\right)\left(1 + zq^{2n+1}\right)\left(1 + z^{-1}q^{2n+1}\right). \tag{6}$$

The second is a result which will be used in Section 3 to prove (5) above. It is the following:

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{\frac{n^2+n}{2}}, \tag{7}$$

where we have used the notation

$$(a;b)_\infty = \prod_{n=1}^{\infty} \left(1 - ab^{n-1}\right).$$

Section 2. The Congruence Involving $\overline{c\phi_2}$.

We begin this section with a theorem concerning the generating function for $\overline{c\phi_2}\,(n)$.

Theorem 2.

$$\sum_{n=0}^{\infty} \overline{c\phi_2}\,(n)\, q^n = \frac{4q\left(q^{16};q^{16}\right)_\infty^2}{(q;q)_\infty^2 \left(q^8;q^8\right)_\infty}.$$

Proof. We know from Kolitsch [3] that

$$\sum_{n=0}^{\infty} \overline{c\phi_2}\,(n)\, q^n = \frac{2}{(q;q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(2n-1)^2}.$$

Hence, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{c\phi_2}\,(n)\,q^n &= \frac{2q}{(q;q)_\infty^2}\sum_{n=-\infty}^{\infty}q^{4n^2-4n}\\ &= \frac{2q}{(q;q)_\infty^2}\prod_{n=1}^{\infty}\left(1-q^{8n}\right)\left(1+q^{8n-8}\right)\left(1+q^{8n}\right)\\ &\qquad\qquad\text{by (6) above}\\ &= \frac{4q}{(q;q)_\infty^2}\left(q^8;q^8\right)_\infty\left(-q^8;q^8\right)_\infty^2\\ &= \frac{4q\left(q^{16};q^{16}\right)_\infty^2}{(q;q)_\infty^2\left(q^8;q^8\right)_\infty}. \quad\blacksquare\end{aligned}$$

The next theorem gives a generating function for $\overline{c\phi_2}\,(2n)$, which will be used to prove (4) above.

Theorem 3.

$$\sum_{n=0}^{\infty}\overline{c\phi_2}\,(2n)\,q^n = \frac{8q\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^4\left(q^4;q^4\right)_\infty}\sum_{n=-\infty}^{\infty}q^{4n^2+5n+1}\sum_{n=-\infty}^{\infty}q^{4n^2+n}.$$

Proof. To prove this, we note that

$$\sum_{n=0}^{\infty}\overline{c\phi_2}\,(2n)\,q^{2n} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\overline{c\phi_2}\,(n)\,q^n+\sum_{n=0}^{\infty}\overline{c\phi_2}\,(n)\,(-q)^n\right].$$

Hence, we know that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{c\phi_2}\,(2n)\,q^{2n} &= \frac{2q\left(q^{16};q^{16}\right)_\infty^2}{(q;q)_\infty^2\left(q^8;q^8\right)_\infty}-\frac{2q\left(q^{16};q^{16}\right)_\infty^2}{(-q;-q)_\infty^2\left(q^8;q^8\right)_\infty}\\ &= \frac{2q\left(q^{16};q^{16}\right)_\infty^2}{\left(q^8;q^8\right)_\infty}\left[\frac{1}{(q;q)_\infty^2}-\frac{1}{(-q;-q)_\infty^2}\right]\\ &= \frac{2q\left(q^{16};q^{16}\right)_\infty^2}{\left(q^2;q^2\right)_\infty^2\left(q^8;q^8\right)_\infty}\left[\frac{1}{\left(q;q^2\right)_\infty^2}-\frac{1}{\left(-q;q^2\right)_\infty^2}\right]\\ &= \frac{2q\left(q^{16};q^{16}\right)_\infty^2}{\left(q^2;q^2\right)_\infty^4\left(q^8;q^8\right)_\infty}\left[\left(q^4;q^4\right)_\infty^2\left(-q;q^2\right)_\infty^2-\left(q^4;q^4\right)_\infty^2\left(q;q^2\right)_\infty^2\right].\end{aligned}$$

Now we focus on the difference above. Notice that it is a difference of two squares and hence can be factored as

$$\left[\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty-\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty\right]\left[\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty+\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty\right].$$

Moreover, from Jacobi's Triple Product Identity,

$$\left(q^4;q^4\right)_\infty \left(-q;q^2\right)_\infty = \sum_{n=-\infty}^{\infty} q^{2n^2+n} \quad \text{and}$$
$$\left(q^4;q^4\right)_\infty \left(q;q^2\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}.$$

Thus,

$$\left(q^4;q^4\right)_\infty \left(-q;q^2\right)_\infty - \left(q^4;q^4\right)_\infty \left(q;q^2\right)_\infty = 2\sum_{n \text{ odd}} q^{2n^2+n} \quad \text{and}$$
$$\left(q^4;q^4\right)_\infty \left(-q;q^2\right)_\infty + \left(q^4;q^4\right)_\infty \left(q;q^2\right)_\infty = 2\sum_{n \text{ even}} q^{2n^2+n}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{c\phi_2}\,(2n)\, q^{2n} &= \frac{8q\left(q^{16};q^{16}\right)_\infty^2}{\left(q^2;q^2\right)_\infty^4 \left(q^8;q^8\right)_\infty} \sum_{n \text{ odd}} q^{2n^2+n} \sum_{n \text{ even}} q^{2n^2+n} \\ &= \frac{8q\left(q^{16};q^{16}\right)_\infty^2}{\left(q^2;q^2\right)_\infty^4 \left(q^8;q^8\right)_\infty} \sum_{n=-\infty}^{\infty} q^{8n^2+10n+3} \sum_{n=-\infty}^{\infty} q^{8n^2+2n} \\ &= \frac{8q^2\left(q^{16};q^{16}\right)_\infty^2}{\left(q^2;q^2\right)_\infty^4 \left(q^8;q^8\right)_\infty} \sum_{n=-\infty}^{\infty} q^{8n^2+10n+2} \sum_{n=-\infty}^{\infty} q^{8n^2+2n}.\end{aligned}$$

Making the replacement $q^2 \to q$ yields the desired result. ■

It is now obvious from Theorem 3 that, for all $n \geq 1$,

$$\overline{c\phi_2}\,(2n) \equiv 0 \pmod{2^3},$$

which proves (4) above.

Section 3. The Congruence Involving $\overline{c\phi_3}$.

We now want to prove the second congruence noted in Theorem 1. We first note the following result, due to Kolitsch [4], which gives the generating function for $\overline{c\phi_3}\,(n)$.

Theorem 4.

$$\sum_{n=0}^{\infty} \overline{c\phi_3}\,(n)\, q^n = \frac{9q\left(q^9;q^9\right)_\infty^3}{\left(q;q\right)_\infty^3 \left(q^3;q^3\right)_\infty}.$$

Now we use an approach quite similar to that used above. Namely, if $\omega = e^{2\pi i/3}$, then

$$\sum_{n=0}^{\infty} \overline{c\phi_3}\,(3n)\, q^{3n} = \frac{1}{3}\left[\sum_{n=0}^{\infty} \overline{c\phi_3}\,(n)\, q^n + \sum_{n=0}^{\infty} \overline{c\phi_3}\,(n)\,(\omega q)^n + \sum_{n=0}^{\infty} \overline{c\phi_3}\,(n)\left(\omega^2 q\right)^n\right].$$

Hence, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{c\phi_3}\,(3n)\,q^{3n} \\
&= \frac{3q\left(q^9;q^9\right)_\infty^3}{\left(q^3;q^3\right)_\infty}\left[\frac{1}{(q;q)_\infty^3}+\frac{\omega}{(\omega q;\omega q)_\infty^3}+\frac{\omega^2}{\left(\omega^2 q;\omega^2 q\right)_\infty^3}\right] \\
&= \frac{3q\left(q^9;q^9\right)_\infty^3}{\left(q^3;q^3\right)_\infty}\left[\frac{(\omega q;\omega q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3+\omega\,(q;q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3+\omega^2\,(q;q)_\infty^3\,(\omega q;\omega q)_\infty^3}{(q;q)_\infty^3\,(\omega q;\omega q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3}\right].
\end{aligned}$$

Now we need to study the sum in brackets above. First, we note that

$$(q;q)_\infty^3\,(\omega q;\omega q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3=\frac{\left(q^3;q^3\right)_\infty^{12}}{\left(q^9;q^9\right)_\infty^3}.$$

(This follows from a straightforward calculation.) Next, we recall from (7) above that

$$\begin{aligned}
(q;q)_\infty^3 &= \sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{\frac{n^2+n}{2}}, \\
(\omega q;\omega q)_\infty^3 &= \sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,\omega^{-n^2-n}q^{\frac{n^2+n}{2}}, \quad \text{and} \\
\left(\omega^2 q;\omega^2 q\right)_\infty^3 &= \sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,\omega^{n^2+n}q^{\frac{n^2+n}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\omega q;\omega q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3+\omega\,(q;q)_\infty^3\left(\omega^2 q;\omega^2 q\right)_\infty^3+\omega^2\,(q;q)_\infty^3\,(\omega q;\omega q)_\infty^3 \\
&\qquad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\,(2m+1)\,(2n+1)\,q^{\frac{m^2+m}{2}+\frac{n^2+n}{2}}\,\Omega
\end{aligned}$$

where

$$\Omega=\omega^{-m^2-m+n^2+n}+\omega^{n^2+n+1}+\omega^{-m^2-m-1}.$$

We now note that many of the terms in this double sum cancel, due to the behavior of Ω and the fact that the sum is symmetric in m and n. This can be seen by taking m and n modulo 3 and calculating Ω. We show these calculations in the table below.

$m \pmod 3$	$n \pmod 3$	Ω
0	0	0
0	1	$1+2\omega^2$
0	2	0
1	0	$1+2\omega$
1	1	3
1	2	$1+2\omega$
2	0	0
2	1	$1+2\omega^2$
2	2	0

Hence, the only contribution occurs when $m \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 3$, and in this case $\Omega = 3$. Therefore,

$$\begin{aligned}(\omega q;\omega q)_\infty^3 \left(\omega^2 q;\omega^2 q\right)_\infty^3 + \omega\,(q;q)_\infty^3 \left(\omega^2 q;\omega^2 q\right)_\infty^3 + \omega^2\,(q;q)_\infty^3\,(\omega q;\omega q)_\infty^3 \\ = 3 \sum_{\substack{m,n\geq 0 \\ m\equiv 1 \pmod 3 \\ n\equiv 1 \pmod 3}} (-1)^{m+n}\,(2m+1)\,(2n+1)\,q^{\frac{m^2+m}{2}+\frac{n^2+n}{2}}.\end{aligned}$$

Combining all of the above remarks, we have the following:

Theorem 5.

$$\sum_{n=0}^{\infty} \overline{c\phi_3}\,(3n)\,q^{3n} = \frac{9q\left(q^9;q^9\right)_\infty^6}{\left(q^3;q^3\right)_\infty^{13}} \sum_{\substack{m,n\geq 0 \\ m\equiv 1 \pmod 3 \\ n\equiv 1 \pmod 3}} (-1)^{m+n}\,(2m+1)\,(2n+1)\,q^{\frac{m^2+m}{2}+\frac{n^2+n}{2}}.$$

Now we note that Theorem 5 implies congruence (5) above. This is easily seen once we realize that, if $m \equiv 1 \pmod 3$, then $2m+1 \equiv 0 \pmod 3$. Therefore, $\overline{c\phi_3}\,(3n)$ is divisible by 81, which is the required result.

Section 4. Final Remarks.

We have now proven that

$$\overline{c\phi_m}\,(mn) \equiv 0 \pmod{m^3}$$

for $m = 2$, 3, 5, 7, and 11. One question that naturally arises is whether congruences of this form occur for larger primes such as $m = 13$ or 17, or for composite values of m. One realistic problem with exploring this is that finding the values for $\overline{c\phi_m}\,(n)$ for large m is not an easy task. These values grow extremely quickly and, consequently, hinder investigation. One step in answering the above question would be to find generating function identities for $\overline{c\phi_m}(n)$ for larger primes m similar to Theorems 3 and 5 above.

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