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RECURRENCES FOR 2–COLORED AND 3–COLORED F–PARTITIONS

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Abstract. The goal of this paper is to prove new recurrences involving 2–colored and 3–colored generalized Frobenius partitions of n similar to the classical recurrence for the partition function $p(n)$.

Section 1. Introduction.

For many years, recurrences for partition functions have been widely used to compute the values of the functions in a straightforward manner. The classical example of a partition function recurrence involves the function $p(n)$:

$$p(n) = \sum_{m\neq 0} (-1)^{m+1} p\left(n - \frac{1}{2}m(3m-1)\right). \tag{1.1}$$

The proof of this result relies heavily on Euler's Pentagonal Number Theorem. (See Andrews [1], Chapter 1, for a complete discussion of this result.)

The goal of this paper is to prove five new recurrences similar to (1.1) involving 2–colored and 3–colored generalized Frobenius partitions, or F–partitions. Namely, we will prove that

$$\overline{c\phi_2}\,(2n) = \sum_{m\neq 0} (-1)^{m+1} \overline{c\phi_2}\left(2n - m^2\right), \tag{1.2}$$

$$\overline{c\phi_2}\,(2n+1) = \left[\sum_{m\neq 0} (-1)^{m+1} \overline{c\phi_2}\left((2n+1) - m^2\right)\right] + 4\sum_{k\geq 0} p(n-2k^2-2k). \tag{1.3}$$

$$\overline{c\phi_3}\,(3n) = \sum_{m\geq 1} (-1)^{m+1}(2m+1) \overline{c\phi_3}\left(3n - \left(\frac{1}{2}m^2 + \frac{1}{2}m\right)\right), \quad \text{and} \tag{1.4}$$

$$\overline{c\phi_3}\,(3n+1) = \left[\sum_{m\geq 1} (-1)^{m+1}(2m+1) \overline{c\phi_3}\left(3n + 1 - \left(\frac{1}{2}m^2 + \frac{1}{2}m\right)\right)\right] + 9a_3(n). \tag{1.5}$$

$$\overline{c\phi_3}\,(3n+2) = \sum_{m\geq 1} (-1)^{m+1}(2m+1) \overline{c\phi_3}\left(3n + 2 - \left(\frac{1}{2}m^2 + \frac{1}{2}m\right)\right) \tag{1.6}$$

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with the convention that $\overline{c\phi_m}\,(n) = 0$ for $n \leq 0$ and $m = 2$ or 3. Here we define $c\phi_m(n)$ as the number of F–partitions of n using m colors [2]. Then $\overline{c\phi_m}\,(n)$ is the number of F–partitions of n using m colors whose order under cyclic permutation of the colors is m. (See [4] and [6] for more on this family of partition functions.) Moreover, $a_3(n)$ is defined as the number of 3–cores of n.

Note the similarity of (1.2), (1.4), and (1.6) to (1.1). The main difference between them is the use of pentagonal numbers in (1.1), squares in (1.2), and triangular numbers in (1.4) and (1.6).

Section 2. One Proof of Recurrence (1.2).

The recurrence (1.2) above will be proven as a corollary of the following key theorem:

Theorem 2.1.

$$\left(\sum_{n\geq 1} \overline{c\phi_2}\,(2n)\, q^{2n}\right)\left(\sum_{m\in\mathbb{Z}} q^{4m^2}\right) = \left(\sum_{n\geq 0} \overline{c\phi_2}\,(2n+1)\, q^{2n+1}\right)\left(\sum_{m\in\mathbb{Z}} q^{(2m+1)^2}\right). \tag{2.1}$$

Proof. The most important tool that will be required in the proof of this theorem is Jacobi's Triple Product Identity:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n\geq 0}\left(1-q^{2n+2}\right)\left(1+zq^{2n+1}\right)\left(1+z^{-1}q^{2n+1}\right) \tag{2.2}$$

Two special cases of (2.2) are given by the following:

$$\sum_{n=-\infty}^{\infty} q^{4n^2} = \sum_{n \text{ even}} q^{n^2} = (q^8;q^8)_\infty(-q^4;q^8)_\infty^2 \tag{2.3}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{(2n+1)^2} = \sum_{n \text{ odd}} q^{n^2} &= q(q^8;q^8)_\infty(-q^8;q^8)_\infty(-q^0;q^8)_\infty \\ &= 2q(q^8;q^8)_\infty(-q^8;q^8)_\infty^2 \\ &= \frac{2q(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\end{aligned} \tag{2.4}$$

where the notation $(a;b)_\infty$ is defined by

$$(a;b)_\infty = \prod_{n=1}^{\infty}(1-ab^{n-1}).$$

From Andrews [2] we see that

$$\sum_{n\geq 0} c\phi_2\,(n)\, q^n = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4(q^4;q^4)_\infty}.$$

Next, since $\overline{c\phi_2}\,(n) = c\phi_2(n) - p(n/2)$, we have

$$\begin{aligned}
\sum_{n\geq 0} \overline{c\phi_2}\,(n)\, q^n &= \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4 (q^4;q^4)_\infty} - \frac{1}{(q^2;q^2)_\infty} \\
&= \left[\frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty^4} - 1\right] \frac{1}{(q^2;q^2)_\infty} \\
&= \left[\frac{(-q;q^2)_\infty^2 - (q;q^2)_\infty^2}{(q;q^2)_\infty^2 (q^2;q^2)_\infty}\right] \frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty} \\
&= \frac{1}{(q;q)_\infty^2} \left[\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right] \quad \text{by (2.2)} \\
&= \frac{2}{(q;q)_\infty^2} \sum_{n \text{ odd}} q^{n^2} \\
&= \frac{4q(q^{16};q^{16})_\infty^2}{(q;q)_\infty^2 (q^8;q^8)_\infty} \quad \text{by (2.4).}
\end{aligned}$$

Now we can develop the generating functions for $\overline{c\phi_2}\,(2n)$ and $\overline{c\phi_2}\,(2n+1)$.

$$\begin{aligned}
\sum_{n\geq 1} \overline{c\phi_2}\,(2n)\, q^{2n} &= \frac{1}{2}\left[\sum_{n\geq 0} \overline{c\phi_2}\,(n)\, q^n + \sum_{n\geq 0} \overline{c\phi_2}\,(n)\, (-q)^n\right] \\
&= \frac{2q(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty} \left[\frac{1}{(q;q^2)_\infty^2} - \frac{1}{(-q;q^2)_\infty^2}\right] \\
&= \frac{2q(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty (q^2;q^4)_\infty^2} \left[(q^2;q^2)_\infty (-q;q^2)_\infty^2 - (q^2;q^2)_\infty (q;q^2)_\infty^2\right] \\
&= \frac{2q(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty (q^2;q^4)_\infty^2} \left(2\sum_{n \text{ odd}} q^{n^2}\right) \quad \text{by (2.2)} \\
&= \frac{2q(q^{16};q^{16})_\infty^2 (q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty} \left[4q(q^8;q^8)_\infty (-q^8;q^8)_\infty^2\right] \\
&= \frac{8q^2(q^{16};q^{16})_\infty^2 (q^4;q^8)_\infty^2 (q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^{16})_\infty^2} \\
&= \frac{8q^2(q^{16};q^{16})_\infty^2 (q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^5 (-q^4;q^8)_\infty^2}. \qquad (2.5)
\end{aligned}$$

Also,

$$
\begin{aligned}
&\sum_{n\geq 0}\overline{c\phi_2}\,(2n+1)q^{2n+1}\\
&\qquad= \frac{1}{2}\left[\sum_{n\geq 0}\overline{c\phi_2}\,(n)\,q^n - \sum_{n\geq 0}\overline{c\phi_2}\,(n)\,(-q)^n\right]\\
&\qquad= \frac{2q(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^2;q^4)_\infty^2}\left[(q^2;q^2)_\infty(-q;q^2)_\infty^2 + (q^2;q^2)_\infty(q;q^2)_\infty^2\right]\\
&\qquad= \frac{2q(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^2;q^4)_\infty^2}\left(2\sum_{n\text{ even}} q^{n^2}\right) \quad \text{by (2.2)}\\
&\qquad= \frac{4q(q^{16};q^{16})_\infty^2(-q^4;q^8)_\infty^2(q^8;q^8)_\infty}{(q^2;q^2)_\infty^3(q^2;q^4)_\infty^2(q^8;q^8)_\infty} \quad \text{by (2.3)}\\
&\qquad= \frac{4q(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^2(q^8;q^{16})_\infty^2}{(q^2;q^2)_\infty^3(q^2;q^4)_\infty^2(q^4;q^8)_\infty^2(q^8;q^8)_\infty^2}\\
&\qquad= \frac{4q(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^5}.
\end{aligned} \tag{2.6}
$$

Multiplying (2.4) by (2.6) and (2.3) by (2.5) we see that

$$
\left(\sum_{n\geq 1}\overline{c\phi_2}\,(2n)\,q^{2n}\right)\left(\sum_{m\in\mathbb{Z}} q^{4m^2}\right) = \left(\sum_{n\geq 0}\overline{c\phi_2}\,(2n+1)\,q^{2n+1}\right)\left(\sum_{m\in\mathbb{Z}} q^{(2m+1)^2}\right)
$$

since both sides equal

$$
\frac{8q^2(q^{16};q^{16})_\infty^2(q^8;q^8)_\infty^3}{(q^2;q^2)_\infty^5}. \quad \blacksquare
$$

Given Theorem 2.1, we can now prove the desired recurrence:

Theorem 2.2. *For all $n\geq 1$,*

$$
\overline{c\phi_2}\,(2n) = \sum_{m\neq 0}(-1)^{m+1}\overline{c\phi_2}\left(2n-m^2\right).
$$

Proof. Theorem 2.1 shows us that

$$
\begin{aligned}
&(\overline{c\phi_2}(2)q^2 + \overline{c\phi_2}(4)q^4 + \overline{c\phi_2}(6)q^6 + \dots)\times\left(1+2q^4+2q^{16}+2q^{36}+2q^{64}+\dots\right) =\\
&\qquad\left(\overline{c\phi_2}(1)q + \overline{c\phi_2}(3)q^3 + \overline{c\phi_2}(5)q^5 + \dots\right)\times\left(2q+2q^9+2q^{25}+2q^{49}+2q^{81}+\dots\right).
\end{aligned}
$$

Comparing the coefficients of q^{2t} on either side of this equality yields

$$
\begin{aligned}
&\overline{c\phi_2}(2t) + 2\overline{c\phi_2}(2t-4) + 2\overline{c\phi_2}(2t-16) + 2\overline{c\phi_2}(2t-36) + 2\overline{c\phi_2}(2t-64) + \dots =\\
&\qquad 2\overline{c\phi_2}(2t-1) + 2\overline{c\phi_2}(2t-9) + 2\overline{c\phi_2}(2t-25) + 2\overline{c\phi_2}(2t-49) + 2\overline{c\phi_2}(2t-81) + \dots
\end{aligned}
$$

Moving everything on the left–hand side over to the right–hand side (except for $\overline{c\phi_2}(2t)$) yields the result of the theorem. $\blacksquare$

Section 3. A Shorter Approach.

Note that (1.2) can also be proven in the following way. Consider

$$\begin{aligned}F(q) := \sum_{n\geq 1} f(n)q^n &= \left(\sum_{n\geq 1}\overline{c\phi_2}(n)q^n\right)\left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)\\ &= \left[\frac{4q(q^{16};q^{16})_\infty^2}{(q;q)_\infty^2(q^8;q^8)_\infty}\right]\left[(q^2;q^2)_\infty(q;q^2)_\infty^2\right]\\ &= \frac{4q(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty(q^2;q^2)_\infty}.\end{aligned}$$

This shows that $F(q)$ is an odd function in q, which means $f(2n)=0$ for all $n\geq 1$. This implies (1.2). We can also prove (1.3) using $F(q)$.

Given that

$$\sum_{n\geq 0} a_2(n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

and

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

we know

$$F(q) = 4\left(\sum_{n\geq 0} a_2(n)q^{8n}\right)\left(\sum_{n\geq 0} p(n)q^{2n+1}\right).$$

Since

$$a_2(n) = \begin{cases} 1, & \text{if } n \text{ is a triangular number,}\\ 0, & \text{otherwise,}\end{cases}$$

we see that $f(2N+1)$ is

$$4\sum_{k\geq 0} p\left(N - 4\left(\frac{1}{2}(k^2+k)\right)\right).$$

Using the same type of argument as above, we then have

$$\overline{c\phi_2}\,(2n+1) = \left[2\sum_{m\neq 0}(-1)^{m+1}\overline{c\phi_2}\left((2n+1)-m^2\right)\right] + 4\sum_{k\geq 0} p(n-2k^2-2k).$$

This is (1.3).

Section 4. The Recurrences for $\overline{c\phi_3}$.

Recurrences (1.4)–(1.6) can be attacked in a similar manner. Consider

$$G(q) := \sum_{n\geq 1} g(n)q^n = \left(\sum_{n\geq 1}\overline{c\phi_3}(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{\frac{n^2+n}{2}}\right).$$

From [5] we know that

$$\sum_{n\geq 1}\overline{c\phi_3}(n)q^n = \frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}$$

and from Jacobi we have

$$\sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{\frac{n^2+n}{2}} = (q;q)_\infty^3.$$

Therefore,

$$\begin{aligned} G(q) &= \left[\frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}\right](q;q)_\infty^3 \\ &= \frac{9q(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}, \end{aligned}$$

which has nonzero power series coefficients only for powers of q of the form q^{3n+1}. Thus, $g(3n)$ and $g(3n-1)$ both equal zero for all $n \geq 1$, and these facts imply (1.4) and (1.6) respectively.

Moreover, since

$$\sum_{n\geq 0} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

we see that

$$G(q) = 9\sum_{n\geq 0} a_3(n)q^{3n+1}.$$

Thus, we have

$$\left(\sum_{n\geq 1}\overline{c\phi_3}(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{\frac{n^2+n}{2}}\right) = 9\sum_{n\geq 0} a_3(n)q^{3n+1}.$$

Considering the terms of the form q^{3n+1} on both sides of this identity yields (1.5).

Section 5. Final Remarks.

Several remarks can be made in closing. First of all, note that (1.2) can be rewritten as

$$\overline{c\phi_2}\,(2n) = 2\sum_{m\geq 1}(-1)^{m+1}\overline{c\phi_2}\left(2n-m^2\right). \tag{5.1}$$

Since it is known from [3] that $\overline{c\phi_2}(n) \equiv 0 \pmod 4$ for all n, (5.1) provides a new proof that $\overline{c\phi_2}(2n) \equiv 0 \pmod 8$. This congruence was proven in [7].

Also, note that (1.2)–(1.6) now allow us to completely determine the values of $\overline{c\phi_2}\,(n)$ and $\overline{c\phi_3}\,(n)$ for all n using recurrences, as long as recurrences exist for $p(n)$ and $a_3(n)$. Indeed, $p(n)$ satisfies one such recurrence thanks to the Pentagonal Number Theorem [1]. It turns out that $a_3(n)$ also satisfies a very nice recurrence, as seen in the following two results.

Theorem 5.1. *If n is not of the form $(3k^2+3k)/2$, then*

$$a_3(n) = a_3(n-1) + a_3(n-2) - a_3(n-5) - a_3(n-7) + a_3(n-12)\dots$$

Note that this is the same recurrence that is satisfied by $p(n)$ for all n.

Proof. The generating function for $a_3(n)$ is given by

$$\sum_{n\geq 0} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

Multiplication by $(q;q)_\infty$ yields the quantity $(q^3;q^3)_\infty^3$ which is equal to

$$\sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{(3n^2+3n)/2}.$$

Since

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2},$$

we see that

$$\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}\right)\left(\sum_{n\geq 0} a_3(n) q^n\right) = \sum_{n=0}^{\infty} (-1)^n \,(2n+1)\, q^{(3n^2+3n)/2}. \tag{5.2}$$

Hence, for $n \neq (3k^2+3k)/2$, the coefficient of q^n on the right–hand side of (5.2) is 0. Thus,

$$a_3(n) - a_3(n-1) - a_3(n-2) + a_3(n-5) + a_3(n-7) - a_3(n-12)\ldots = 0.$$

This is equivalent to the result in Theorem 5.1. ∎

Now we simply must ask whether a recurrence exists for $a_3(n)$ where $n = (3k^2+3k)/2$. Indeed, such a recurrence exists.

Theorem 5.2. *If* $n = (3k^2+3k)/2$*, then*

$$a_3(n) = [a_3(n-1) + a_3(n-2) - a_3(n-5) - a_3(n-7) + a_3(n-12)\ldots] + (-1)^k(2k+1). \tag{5.3}$$

Proof. Again returning to (5.2) above, we see that comparison of the coefficient of $q^{(3k^2+3k)/2}$ on each side yields

$$a_3(n) - a_3(n-1) - a_3(n-2) + a_3(n-5) + a_3(n-7) - a_3(n-12)\cdots = (-1)^k(2k+1).$$

Moving all the terms to the right–hand side except for $a_3(n)$ yields (5.3). ∎

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