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# ELEMENTARY PROOFS OF INFINITELY MANY CONGRUENCES FOR 8–CORES

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Abstract. Using a very elementary argument, we prove the congruences

$$a_8(81n + 21) \equiv 0 \pmod 2 \quad \text{and}$$
$$a_8(81n + 75) \equiv 0 \pmod 2$$

where $a_8(n)$ is the number of 8–core partitions of $n$. We also exhibit two infinite families of congruences modulo 2 for 8–cores.

## 1. Background

For a positive integer $n$, we let $a_t(n)$ be the number of partitions of $n$ whose Ferrers graphs are void of hooks with lengths that are multiples of $t$ [5]. We say $a_t(n)$ is the number of $t$–core partitions, or $t$–cores, of $n$.

A number of congruences for the functions $a_t(n)$ have been proven recently. For example, the congruences

$$a_5(5^\alpha n - 1) \equiv 0 \pmod{5^\alpha} \text{ and}$$
$$a_7(7^\alpha n - 2) \equiv 0 \pmod{7^\alpha}$$

are proven in [2] and are clearly related to congruences proven for $p(n)$ by Ramanujan [9]. See, for example, [1], [6], and [7] for additional work on $t$–cores.

More recently, attention has been paid to $t$–cores where $t$ is a power of 2. Congruences for $a_2(n)$ are somewhat trivial since

$$a_2(n) = \begin{cases} 1, & \text{if } n \text{ is a triangular number} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

So the first interesting case to study is $a_4(n)$. Much has been said about this function, which has a rich underlying structure. Hirschhorn and Sellers [3], [4] have proven infinitely many congruences for 4–cores which can be derived from the following arithmetic identities:

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For $\lambda \geq 1$,

$$\begin{aligned} a_4(3^{2\lambda+1}n + (5 \times 3^{2\lambda} - 5)/8) &= 3^\lambda \times a_4(3n), \\ a_4(3^{2\lambda+1}n + (13 \times 3^{2\lambda} - 5)/8) &= (2 \times 3^\lambda - 1) \times a_4(3n+1), \\ a_4(3^{2\lambda+2}n + (7 \times 3^{2\lambda+1} - 5)/8) &= ((3^{\lambda+1} - 1)/2) \times a_4(9n+2), \quad \text{and} \\ a_4(3^{2\lambda+2}n + (23 \times 3^{2\lambda+1} - 5)/8) &= ((3^{\lambda+1} - 1)/2) \times a_4(9n+8). \end{aligned}$$

For additional discussion on 4–cores, see [8].

## 2. Our Results

The goal of this paper is to prove the following facts for 8–cores:

**Theorem.** *For all $n \geq 0$,*

$$\begin{aligned} a_8(81n+21) &\equiv 0 \pmod{2} \quad \textit{and} \\ a_8(81n+75) &\equiv 0 \pmod{2}. \end{aligned}$$

*Proof.* From [2], the generating function for $a_8(n)$ is

$$\sum_{n \geq 0} a_8(n) q^n = \frac{(q^8;q^8)_\infty^8}{(q;q)_\infty}$$

where $(a;b)_\infty = (1-a)(1-ab)(1-ab^2)(1-ab^3)\dots$

Thus,

$$\sum_{n \geq 0} a_8(n) q^n = \frac{(q^8;q^8)_\infty^8}{(q^4;q^4)_\infty^4} \cdot \frac{(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^2} \cdot \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

We write the generating function in this fashion to make use of the fact that

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \sum_{k>0} q^{(k^2-k)/2},$$

which is the generating function version of (1) above. Therefore,

$$\sum_{n \geq 0} a_8(n) q^{8n+21} \equiv \left(\sum_{k>0} q^{16(2k-1)^2}\right)\left(\sum_{k>0} q^{4(2k-1)^2}\right)\left(\sum_{k>0} q^{(2k-1)^2}\right) \pmod{2}.$$

Since the odd squares modulo 6 are congruent to 1 or 3, we see that $a_8(27n+21)$ is congruent (modulo 2) to the number of representations of $216n+189$ as either $(6r \pm 1)^2 + 4(6s \pm 1)^2 + 16(6t \pm 1)^2$ or $(6r-3)^2 + 4(6s-3)^2 + 16(6t-3)^2$. (No other combination of squares congruent to 1 or 3 modulo 6 will yield an appropriate value.)

Note also the following:

$$\begin{aligned}
216n+189 &= (6r\pm 1)^2+4(6s\pm 1)^2+16(6t\pm 1)^2 \\
&\text{iff } 18n+14 = (3r^2\pm r)+4(3s^2\pm s)+16(3t^2\pm t), \quad \text{and} \qquad (2)\\
216n+189 &= (6r-3)^2+(6s-3)^2+(6t-3)^2 \\
&\text{iff } 24n+21 = (2r-1)^2+4(2s-1)^2+16(2t-1)^2 \\
&\text{iff } 2n = (3R^2\pm R)+4(3S^2\pm S)+16(3T^2\pm T) \\
&\text{or } 2n-14 = 3(R^2-R)+12(S^2-S)+48(T^2-T) \qquad (3)
\end{aligned}$$

Now we recall two well–known facts from the theory of partitions. The first is Euler's Pentagonal Number Theorem, which states that

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2+k)/2}.$$

The second is a fact due to Jacobi:

$$(q;q)_\infty^3 = \sum_{k\geq 0} (-1)^k (2k+1) q^{(k^2+k)/2}$$

Clearly, these two imply

$$\begin{aligned}
(q;q)_\infty &\equiv \sum_{k=-\infty}^{\infty} q^{(3k^2+k)/2} \pmod 2 \text{ and} \\
(q;q)_\infty^3 &\equiv \sum_{k\geq 1} q^{k^2-k} \pmod 2. \qquad (4)
\end{aligned}$$

Thanks to the reductions (2) and (3) involving $18n+14$ and $2n$ above, we can now state that $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{9n+7}$ in

$$\begin{aligned}
&(q;q)_\infty(q^4;q^4)_\infty(q^{16};q^{16})_\infty + q^7(q^9;q^9)_\infty(q^{36};q^{36})_\infty(q^{144};q^{144})_\infty \\
&\qquad + q^{70}(q^{27};q^{27})_\infty^3(q^{108};q^{108})_\infty^3(q^{432};q^{432})_\infty^3.
\end{aligned}$$

Since

$$\begin{aligned}
(q;q)_\infty(q^4;q^4)_\infty(q^{16};q^{16})_\infty &\equiv (q;q)_\infty(q;q)_\infty^4(q;q)_\infty^{16} \pmod 2 \\
&= (q;q)_\infty^3(q;q)_\infty^6(q;q)_\infty^{12} \\
&\equiv (q;q)_\infty^3(q^2;q^2)_\infty^3(q^4;q^4)_\infty^3 \pmod 2,
\end{aligned}$$

we can actually say that $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{9n+7}$ in

$$\begin{aligned}
&(q;q)_\infty^3(q^2;q^2)_\infty^3(q^4;q^4)_\infty^3 + q^7(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3(q^{36};q^{36})_\infty^3 \\
&\qquad + q^{70}(q^{27};q^{27})_\infty^3(q^{108};q^{108})_\infty^3(q^{432};q^{432})_\infty^3.
\end{aligned}$$

Now by using (4) and completing the square we know that $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{72n+63}$ in

$$\left(\sum_{m>0} q^{4(2m-1)^2}\right)\left(\sum_{m>0} q^{2(2m-1)^2}\right)\left(\sum_{m>0} q^{(2m-1)^2}\right)+ \\ +\left(\sum_{m>0} q^{36(2m-1)^2}\right)\left(\sum_{m>0} q^{18(2m-1)^2}\right)\left(\sum_{m>0} q^{9(2m-1)^2}\right) \\ +q^{567}(q^{216};q^{216})_\infty^3(q^{864};q^{864})_\infty^3(q^{3456};q^{3456})_\infty^3.$$

Again considering that odd squares are always congruent to 1 or 3 modulo 6, we determine that $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{72n+63}$ in

$$\left(\sum q^{4(6m\pm1)^2}\right)\left(\sum q^{2(6m\pm1)^2}\right)\left(\sum q^{(6m-3)^2}\right) \\ +\left(\sum q^{4(6m-3)^2}\right)\left(\sum q^{2(6m\pm1)^2}\right)\left(\sum q^{(6m\pm1)^2}\right) \\ +\left(\sum q^{4(6m-3)^2}\right)\left(\sum q^{2(6m-3)^2}\right)\left(\sum q^{(6m-3)^2}\right) \\ +\left(\sum q^{4(6m-3)^2}\right)\left(\sum q^{2(6m-3)^2}\right)\left(\sum q^{(6m-3)^2}\right) \\ +q^{567}(q^{216};q^{216})_\infty^3(q^{864};q^{864})_\infty^3(q^{3456};q^{3456})_\infty^3.$$

Using reduction processes similar to those noted in (2) and (3) we see that $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{3n+2}$ in

$$(q^4;q^4)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty^3+q(q^{12};q^{12})_\infty^3(q^2;q^2)_\infty(q;q)_\infty+q^{23}(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty^3(q^{144};q^{144})_\infty^3,$$

which is congruent modulo 2 to

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3+q(q^{12};q^{12})_\infty^3(q;q)_\infty^3+q^{23}(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty^3(q^{144};q^{144})_\infty^3. \qquad (5)$$

Once more, we complete the square. Then $a_8(27n+21)$ is congruent (modulo 2) to the coefficient of $q^{24n+21}$ in

$$\left(\sum q^{2(6m-3)^2}\right)\left(\sum q^{3(6m\pm1)^2}\right) \\ +\left(\sum q^{2(6m-3)^2}\right)\left(\sum q^{3(6m-3)^2}\right) \\ +\left(\sum q^{12(6m\pm1)^2}\right)\left(\sum q^{(6m-3)^2}\right) \\ +\left(\sum q^{12(6m-3)^2}\right)\left(\sum q^{(6m-3)^2}\right) \\ +q^{189}(q^{72};q^{72})_\infty^3(q^{288};q^{288})_\infty^3(q^{1152};q^{1152})_\infty^3.$$

One last set of reductions leads to a very nice result. Namely,

$$\begin{aligned}&\sum_{n\geq 0} a_8(27n+21)q^n\\ &\equiv (q^6;q^6)_\infty^3(q^3;q^3)_\infty + q(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3 + (q^{12};q^{12})_\infty(q^3;q^3)_\infty^3\\ &\quad + q^4(q^{36};q^{36})_\infty^3(q^3;q^3)_\infty^3 + q^7(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty^3(q^{48};q^{48})_\infty^3 \pmod 2\\ &\equiv q(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3 + q^4(q^{36};q^{36})_\infty^3(q^3;q^3)_\infty^3 + q^7(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty^3(q^{48};q^{48})_\infty^3 \pmod 2\end{aligned} \tag{6}$$

since

$$(q^6;q^6)_\infty^3(q^3;q^3)_\infty + (q^{12};q^{12})_\infty(q^3;q^3)_\infty^3 \equiv 2(q^3;q^3)_\infty^7 \pmod 2.$$

This is now extremely helpful in proving the congruences in our theorem. We can note that in all of the terms in this last generating function the power of $q$ is congruent to 1 modulo 3. Thus, $a_8(81n+21) \equiv 0 \pmod 2$ and $a_8(81n+75) \equiv 0 \pmod 2$. □

We now consider $a_8(81n+48)$, which is congruent to the coefficient of $q^n$ in

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3 + q(q^{12};q^{12})_\infty^3(q;q)_\infty^3 + q^2(q;q)_\infty^3(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty^3 \pmod 2,$$

which is congruent to

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3 + q(q^{12};q^{12})_\infty^3(q;q)_\infty^3 + \sum_{n\geq 0} a_8(n)q^{n+2} \pmod 2.$$

Thus, $a_8(243n+210)$ is congruent modulo 2 to the coefficient of $q^{3n+2}$ in

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3 + q(q^{12};q^{12})_\infty^3(q;q)_\infty^3 + \sum_{n\geq 0} a_8(n)q^{n+2}.$$

From (5) and (6) we see that $a_8(243n+210)$ is congruent modulo 2 to the coefficient of $q^n$ in

$$q(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3 + q^4(q^{36};q^{36})_\infty^3(q^3;q^3)_\infty^3 + \sum_{n\geq 0} a_8(3n)q^n.$$

From this we see that $a_8(729n+210) \equiv a_8(9n) \pmod 2$ and $a_8(729n+696) \equiv a_8(9n+6) \pmod 2$. Now $a_8(729n+453)$ is congruent modulo 2 to the coefficient of $q^n$ in

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3 + q(q^{12};q^{12})_\infty^3(q;q)_\infty^3 + \sum_{n\geq 0} a_8(9n+3)q^n.$$

Thus, $a_8(2187n+1911)$ is congruent to the coefficient of $q^n$ in

$$q(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3 + q^4(q^{36};q^{36})_\infty^3(q^3;q^3)_\infty^3 + \sum_{n\geq 0} a_8(27n+21)q^n.$$

Hence, $a_8(6561n+1911) \equiv a_8(81n+21) \pmod 2$ and $a_8(6561n+6285) \equiv a_8(81n+75) \pmod 2$. In addition, $a_8(6561n+4098)$ is congruent modulo 2 to the coefficient of $q^n$ in

$$(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3 + q(q^{12};q^{12})_\infty^3(q;q)_\infty^3 + \sum_{n\geq 0} a_8(81n+48)q^n,$$

which is congruent modulo 2 to $\sum_{n\geq 0} a_8(n)q^{n+2}$. Therefore, $a_8(6561n+4098) \equiv a_8(n-2) \pmod 2$. This proves the following theorem:

**Theorem.** *For all $n \geq 0$,*

$$a_8(81^\alpha n + \lambda_\alpha) \equiv 0 \pmod 2 \quad \textit{and}$$
$$a_8(81^\alpha n + \beta_\alpha) \equiv 0 \pmod 2$$

*where* $\lambda_\alpha = (189 \cdot 81^{\alpha-1} - 21)/8$ *and* $\beta_\alpha = (621 \cdot 81^{\alpha-1} - 21)/8$.

## 3. Final Thought

Because congruences in arithmetic progressions for $t$–cores appear to become quite sparse as $t$ increases, it is surprising to find such nice divisibility results. Moreover, it is satisfying to find such an elementary sieving argument to prove these easily.

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