Arithmetic Properties of Partitions with Even Parts Distinct

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Abstract

In this work, we consider the function $ped(n)$, the number of partitions of an integer n wherein the even parts are distinct (and the odd parts are unrestricted). Our goal is to consider this function from an arithmetical point of view in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. We prove a number of results for $ped(n)$ including the following: For all $n \geq 0$,

$$ped(9n+4) \equiv 0 \pmod 4$$

and

$$ped(9n+7) \equiv 0 \pmod{12}.$$

Indeed, we compute appropriate generating functions from which we deduce these congruences and find, in particular, the surprising result that

$$\sum_{n\geq 0} ped(9n+7)q^n = 12\frac{(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^6 (q^4;q^4)_\infty}{(q;q)_\infty^{11}}.$$

We also show that $ped(n)$ is divisible by 6 at least 1/6 of the time.

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1 Introduction

In recent years, the function which enumerates those integer partitions wherein even parts are distinct (and odd parts are unrestricted) has arisen quite naturally. For example, the generating function for these partitions appears in the following classic identity of Lebesgue [8]:

$$\sum_{n\geq 0}\left(\prod_{i=1}^{n}\frac{1+q^i}{1-q^i}\right) q^{\binom{n+1}{2}} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}$$

where $(a;q)_m = \prod_{n=0}^{m-1}(1-aq^n)$ and $(a;q)_\infty = \lim_{m\to\infty}(a;q)_m$.

As the above equation shows, the number of partitions of n wherein even parts are distinct equals the number of partitions of n with no parts divisible by 4. These are often referred to as 4–regular partitions and much has been written about arithmetic properties of such partitions. The reader interested in work involving regular partitions may wish to see Alladi [1], Andrews [2, Theorem 9], [3], Dandurand and Penniston [5], Granville and Ono [6], Gordon and Ono [7], Patkowski [9], and Penniston [10].

Our goal in this work is to consider $ped(n)$, the number of partitions of n wherein the even parts are distinct, from an arithmetic point of view in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. In this vein, we will prove various congruence properties satisfied by $ped(n)$ as well as a number of explicit results on generating function dissections. In particular, we will prove that

$$\sum_{n\geq 0} ped(9n+1)q^n = \frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^4(q^4;q^4)_\infty}{(q;q)_\infty^5(q^6;q^6)_\infty^2} + 24q\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3(q^4;q^4)_\infty(q^6;q^6)_\infty^3}{(q;q)_\infty^{10}},$$

$$\sum_{n\geq 0} ped(9n+4)q^n = 4\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^4} + 48q\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^6}{(q;q)_\infty^9}$$

and

$$\sum_{n\geq 0} ped(9n+7)q^n = 12\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^6(q^4;q^4)_\infty}{(q;q)_\infty^{11}}$$

from which it is immediate that for $n \geq 0$,

$$ped(9n+4) \equiv 0 \pmod 4$$

and

$$ped(9n+7) \equiv 0 \pmod{12}.$$

We also deduce that for $\alpha \geq 1$ and all $n \geq 0$,

$$ped\left(3^{2\alpha+1}n + \frac{17\times 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod 6,$$

$$ped\left(3^{2\alpha+2}n + \frac{19\times 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 6$$

and that $ped(n)$ is divisible by 6 at least 1/6 of the time.

In the proofs below, we use nothing deeper than Ramanujan's ${}_1\psi_1$ summation formula, which as noted in [4, Theorem 10.5.1], is given as follows: For $|q| < 1$ and $|b/a| < |x| < 1$,

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n}x^n = \frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(x;q)_\infty(b/ax;q)_\infty(b;q)_\infty(q/a;q)_\infty}. \tag{1}$$

2 Preliminaries

We shall require several properties of the functions denoted $\phi(q)$ and $\psi(q)$ by Ramanujan, namely

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{(n^2+n)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2-n}.$$

The necessary properties are given in the following lemmas. We include proofs for the sake of completeness.

Lemma 2.1. $\phi(-q) = \dfrac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$

Proof.

$$\begin{aligned}
\phi(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \\
&= (q;q^2)_\infty^2 (q^2;q^2)_\infty \quad \text{by Jacobi's Triple Product Identity} \\
&= (q;q)_\infty (q;q^2)_\infty \\
&= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.
\end{aligned}$$

□

Lemma 2.2. $\psi(-q) = \dfrac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$

Proof. We have

$$\begin{aligned}
\psi(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n} \\
&= (q;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty \quad \text{by Jacobi's Triple Product Identity} \\
&= \frac{(q;q)_\infty}{(q^2;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty (q;q^2)_\infty}{(q^2;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.
\end{aligned}$$

□

Lemma 2.3. $\phi(-q)\psi(-q) = \dfrac{(q;q)_\infty (q^2;q^2)_\infty}{(-q;q)_\infty (-q;q^2)_\infty}$

Proof. We have

$$\begin{aligned}
\phi(-q)\psi(-q) &= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \quad \text{by Lemmas 2.1 and 2.2} \\
&= \frac{(q;q)_\infty^2}{(-q;q^2)_\infty} \\
&= \frac{(q;q)_\infty (q^2;q^2)_\infty}{(-q;q)_\infty (-q;q^2)_\infty}.
\end{aligned}$$

□

Lemma 2.4. $\dfrac{\phi(-q)}{\psi(-q)} = \dfrac{(q;q^2)_\infty}{(-q^2;q^2)_\infty}$

Proof. We have

$$\begin{aligned}
\frac{\phi(-q)}{\psi(-q)} &= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \quad \text{by Lemmas 2.1 and 2.2} \\
&= \frac{(-q;q^2)_\infty}{(-q;q)_\infty^2} \\
&= \frac{1}{(-q;q)_\infty(-q^2;q^2)_\infty} \\
&= \frac{(q;q)_\infty}{(q^2;q^2)_\infty(-q^2;q^2)_\infty} \\
&= \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty}.
\end{aligned}$$

□

We shall also need the following results.

Lemma 2.5. $\dfrac{(q;q)_\infty^2(ax;q)_\infty(q/ax;q)_\infty}{(a;q)_\infty(q/a;q)_\infty(x;q)_\infty(q/x;q)_\infty} = \displaystyle\sum_{n=-\infty}^{\infty}\frac{x^n}{1-aq^n}.$

Proof. In Ramanujan's ${}_1\psi_1$ summation formula 1 set $b = aq$, then divide by $1-a$. □

Lemma 2.6. *$\phi(-q) = a(q^3) - 2qb(q^3)$ where $a(q) = \phi(-q^3)$ and $b(q) = \dfrac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}$.*

Proof.

$$\begin{aligned}
\phi(-q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} \\
&= \sum_{n=-\infty}^{\infty}(-1)^{3n}q^{(3n)^2} + \sum_{n=-\infty}^{\infty}(-1)^{3n-1}q^{(3n-1)^2} + \sum_{n=-\infty}^{\infty}(-1)^{3n+1}q^{(3n+1)^2} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2} - 2q\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2-6n} \\
&= a(q^3) - 2qb(q^3)
\end{aligned}$$

where

$$a(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2} = \phi(-q^3)$$

and

$$\begin{aligned}
b(q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} \\
&= (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty \quad \text{by Jacobi's Triple Product Identity} \\
&= \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}.
\end{aligned}$$

□

Lemma 2.7. $a(q)^3 - 8qb(q)^3 = \dfrac{\phi(-q)^4}{\phi(-q^3)}$

Proof. First,

$$\begin{aligned}
(q;q)_\infty(\omega q;\omega q)_\infty(\omega^2 q;\omega^2 q)_\infty &= \prod_{n=1}^{\infty}(1-q^n)(1-\omega^n q^n)(1-\omega^{2n}q^n) \\
&= \prod_{3\,|\,n}(1-q^n)^3\prod_{3\nmid n}(1-q^{3n}) \\
&= \frac{\prod_{n\geq 1}(1-q^{3n})^4}{\prod_{3\,|\,n}(1-q^{3n})} \\
&= \frac{\prod_{n\geq 1}(1-q^{3n})^4}{\prod_{n\geq 1}(1-q^{9n})} \\
&= \frac{(q^3;q^3)_\infty^4}{(q^9;q^9)_\infty}.
\end{aligned}$$

Next,

$$\begin{aligned}
\phi(-q)\phi(-\omega q)\phi(-\omega^2 q) &= \frac{(q;q)_\infty^2(\omega q;\omega q)_\infty^2(\omega^2 q;\omega^2 q)_\infty^2}{(q^2;q^2)_\infty(\omega^2 q^2;\omega^2 q^2)_\infty(\omega q^2;\omega q^2)_\infty} \quad \text{by Lemma 2.1} \\
&= \left(\frac{(q^3;q^3)_\infty^4}{(q^9;q^9)_\infty}\right)^2\frac{(q^{18};q^{18})_\infty}{(q^6;q^6)_\infty^4} \\
&= \left(\frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}\right)^4\frac{(q^{18};q^{18})_\infty}{(q^9;q^9)_\infty^2} \\
&= \frac{\phi(-q^3)^4}{\phi(-q^9)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
a(q^3)^3-8q^3b(q^3)^3 &= (a(q^3)-2qb(q^3))(a(q^3)-2\omega qb(q^3))(a(q^3)-2\omega^2 qb(q^3)) \\
&= \phi(-q)\phi(-\omega q)\phi(-\omega^2 q) \\
&= \frac{\phi(-q^3)^4}{\phi(-q^9)}.
\end{aligned}$$

If we replace q^3 by q, we obtain the required result. □

Lemma 2.8.

$$\frac{1}{(a-2qb)^2}=\frac{a^4+4qa^3b+12q^2a^2b^2+16q^3ab^3+16q^4b^4}{(a^3-8q^3b^3)^2}$$

Remark 2.9. *Note that this is an elementary lemma - it holds for all a and b provided $a-2qb\neq 0$.*

Proof.

$$\begin{aligned}
\frac{1}{(a-2qb)^2} &= \left(\frac{a^2+2qab+4q^2b^2}{a^3-8q^3b^3}\right)^2 \\
&= \frac{a^4+4qa^3b+12q^2a^2b^2+16q^3ab^3+16q^4b^4}{(a^3-8q^3b^3)^2}.
\end{aligned}$$

□

With the above lemmas proved, we can now move to the dissections of the generating function for $ped(n)$ with the goal of proving the desired congruences.

3 Main Results

We shall begin by proving the following theorem which provides our first set of important dissections.

Theorem 3.1.

$$\begin{aligned}\sum_{n\geq 0} ped(3n)q^n &= \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^4}{(q;q)_\infty^3 (q^{12};q^{12})_\infty^2},\\ \sum_{n\geq 0} ped(3n+1)q^n &= \frac{\phi(-q^3)\psi(-q^3)}{\phi(-q)^2}, \quad \textit{and}\\ \sum_{n\geq 0} ped(3n+2)q^n &= 2\frac{(q^2;q^2)_\infty (q^6;q^6)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^3}.\end{aligned}$$

Proof.

$$\begin{aligned}\sum_{n\geq 0} ped(n)q^n &= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\\ &= \frac{(-q^2;q^6)_\infty(-q^4;q^6)_\infty(-q^6;q^6)_\infty}{(q;q^6)_\infty(q^3;q^6)_\infty(q^5;q^6)_\infty}\\ &= \frac{(-q^6;q^6)_\infty(-q^3;q^6)_\infty^2}{(q^3;q^6)_\infty(q^6;q^6)_\infty^2}\left(\frac{(q^6;q^6)_\infty^2(-q^2;q^6)_\infty(-q^4;q^6)_\infty}{(-q^3;q^6)_\infty^2(q;q^6)_\infty(q^5;q^6)_\infty}\right)\\ &= \frac{(-q^3;q^3)_\infty(-q^3;q^6)_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{6n+3}} \quad \text{by Lemma 2.5}\\ &= \frac{1}{\phi(-q^3)\psi(-q^3)}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{6n+3}} \quad \text{by Lemma 2.3.}\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} ped(3n)q^n &= \frac{1}{\phi(-q)\psi(-q)}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{6n+1}}\\ &= \frac{1}{\phi(-q)\psi(-q)}\left(\frac{(q^6;q^6)_\infty^2(-q^2;q^6)_\infty(-q^4;q^6)_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}\right)\\ &= \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^4}{(q;q)_\infty^3(q^{12};q^{12})_\infty^2}\end{aligned}$$

after simplification, using Lemmas 2.1, 2.2 and such devices as

$$(-q^2;q^6)_\infty(-q^4;q^6)_\infty = \frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}, \quad (-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty},$$

$$(-q;q^2)_\infty = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty} \quad \text{and} \quad (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}.$$

Similarly,

$$\begin{aligned}\sum_{n\geq 0} ped(3n+2)q^n &= \frac{1}{\phi(-q)\psi(-q)}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{6n+5}}\\ &= \frac{1}{\phi(-q)\psi(-q)}\left(\frac{(q^6;q^6)_\infty^2(-q^6;q^6)_\infty(-1;q^6)_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}\right)\\ &= \frac{2}{\phi(-q)\psi(-q)}\left(\frac{(q^6;q^6)_\infty^2(-q^6;q^6)_\infty(-q^6;q^6)_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}\right)\\ &= 2\frac{(q^2;q^2)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3}\end{aligned}$$

again after simplification. Lastly,

$$\begin{aligned}
\sum_{n\geq 0} ped(3n+1)q^n &= \frac{1}{\phi(-q)\psi(-q)}\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{6n+3}} \\
&= \frac{\phi(-q^3)\psi(-q^3)}{\phi(-q)\psi(-q)}\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \quad \text{from the above work} \\
&= \frac{\phi(-q^3)\psi(-q^3)}{\phi(-q)^2} \quad \text{by Lemma 2.4.}
\end{aligned}$$

□

With the above set of dissections complete, we now move to a second set of dissections which provide the proofs of the desired congruences. The main results are summarized in the following theorem:

Theorem 3.2.

$$\begin{aligned}
\sum_{n\geq 0} ped(9n+1)q^n &= \frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^4(q^4;q^4)_\infty}{(q;q)_\infty^5(q^6;q^6)_\infty^2} + 24q\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3(q^4;q^4)_\infty(q^6;q^6)_\infty^3}{(q;q)_\infty^{10}}, \\
\sum_{n\geq 0} ped(9n+4)q^n &= 4\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^4} + 48q\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^6}{(q;q)_\infty^9}, \\
\sum_{n\geq 0} ped(9n+7)q^n &= 12\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^6(q^4;q^4)_\infty}{(q;q)_\infty^{11}}.
\end{aligned}$$

Proof. By Lemmas 2.6 and 2.8, we have

$$\begin{aligned}
&\sum_{n\geq 0} ped(3n+1)q^n \\
= &\ \frac{\phi(-q^3)\psi(-q^3)}{\phi(-q)^2} \\
= &\ \frac{\phi(-q^3)\psi(-q^3)}{(a(q^3)-2qb(q^3))^2} \\
= &\ \phi(-q^3)\psi(-q^3) \\
&\times\frac{a(q^3)^4+4qa(q^3)^3b(q^3)+12q^2a(q^3)^2b(q^3)^2+16q^3a(q^3)b(q^3)^3+16q^4b(q^3)^4}{(a(q^3)^3-8q^3b(q^3)^3)^2}.
\end{aligned}$$

From Lemmas 2.6 and 2.7, it follows that

$$\begin{aligned}
\sum_{n\geq 0} ped(9n+1)q^n &= \phi(-q)\psi(-q)\frac{a(q)^4+16qa(q)b(q)^3}{(a(q)^3-8qb(q)^3)^2} \\
&= \frac{\psi(-q)\phi(-q^3)^3}{\phi(-q)^7}\left(a(q)^3+16qb(q)^3\right) \\
&= \frac{\psi(-q)\phi(-q^3)^3}{\phi(-q)^7}\left((a(q)^3-8qb(q)^3)+24qb(q)^3\right) \\
&= \frac{\psi(-q)\phi(-q^3)^3}{\phi(-q)^7}\left(\frac{\phi(-q)^4}{\phi(-q^3)}+24q\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)^3\right) \\
&= \frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^4(q^4;q^4)_\infty}{(q;q)_\infty^5(q^6;q^6)_\infty^2} + 24q\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^3(q^4;q^4)_\infty(q^6;q^6)_\infty^3}{(q;q)_\infty^{10}}
\end{aligned}$$

after simplification. Similarly,

$$\begin{aligned}
&\sum_{n\ge 0} ped(9n+4)q^n \\
=\ & \phi(-q)\psi(-q)\frac{4a(q)^3b(q)+16qb(q)^4}{(a(q)^3-8qb(q)^3)^2} \\
=\ & 4\frac{\psi(-q)\phi(-q^3)^2}{\phi(-q)^7}\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)\left(a(q)^3+4qb(q)^3\right) \\
=\ & 4\frac{\psi(-q)\phi(-q^3)^2}{\phi(-q)^7}\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)\left((a(q)^3-8qb(q)^3)+12qb(q)^3\right) \\
=\ & 4\frac{\psi(-q)\phi(-q^3)^2}{\phi(-q)^7}\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)\left(\frac{\phi(-q)^4}{\phi(-q^3)}+12q\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)^3\right) \\
=\ & 4\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^4}+48q\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^6}{(q;q)_\infty^9}
\end{aligned}$$

after simplification. Lastly,

$$\begin{aligned}
\sum_{n\ge 0} ped(9n+7)q^n &= \phi(-q)\psi(-q)\frac{12a(q)^2b(q)^2}{(a(q)^3-8qb(q)^3)^2} \\
&= 12\frac{\psi(-q)\phi(-q^3)^4}{\phi(-q)^7}\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)^2 \\
&= 12\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^6(q^4;q^4)_\infty}{(q;q)_\infty^{11}}.
\end{aligned}$$

□

Corollary 3.3. *For all* $n \ge 0$,

$$\begin{aligned}
ped(9n+4) &\equiv 0 \pmod{4}, \\
\text{and}\ \ ped(9n+7) &\equiv 0 \pmod{12}.
\end{aligned}$$

Proof. These congruences are immediate from Theorem 2.2. □

With these results in hand, we now wish to prove two infinite families of Ramanujan–like congruences modulo 6 satisfied by $ped(n)$. We begin by proving the following:

Theorem 3.4. *For* $\alpha \ge 1$,

$$\begin{aligned}
\sum_{n\ge 0} ped\left(3^{2\alpha}n+\frac{3^{2\alpha}-1}{8}\right)q^n &\equiv \psi(q) \pmod{2}, \\
\sum_{n\ge 0} ped\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{8}\right)q^n &\equiv \psi(q^3) \pmod{2}, \\
\sum_{n\ge 0} ped\left(3^{2\alpha}n+\frac{3^{2\alpha}-1}{8}\right)q^n &\equiv (-1)^{\alpha-1}\psi(-q)\phi(-q^3) \pmod{3}, \quad \textit{and} \\
\sum_{n\ge 0} ped\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{8}\right)q^n &\equiv (-1)^{\alpha}\phi(-q)\psi(-q^3) \pmod{3}.
\end{aligned}$$

Proof. From the work above, we have

$$\sum_{n\geq 0} ped(9n+1)q^n = \frac{\psi(-q)\phi(-q^3)^3}{\phi(-q)^7}\left(\frac{\phi(-q)^4}{\phi(-q^3)} + 24q\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right)^3\right).$$

It follows that, modulo 2,

$$\sum_{n\geq 0} ped(9n+1)q^n \equiv \frac{\psi(-q)\phi(-q^3)^2}{\phi(-q)^3} \equiv \psi(-q) \equiv \psi(q) = f(q^3,q^6) + q\psi(q^9)$$

where $f(a,b)$ is Ramanujan's theta–function,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{(n^2+n)/2}b^{(n^2-n)/2}.$$

The mod 2 result now follows by induction on α. Similarly, modulo 3,

$$\sum_{n\geq 0} ped(9n+1)q^n \equiv \frac{\psi(-q)\phi(-q^3)^2}{\phi(-q)^3} \equiv \psi(-q)\phi(-q^3) = \phi(-q^3)\left(f(-q^3,q^6) - q\psi(-q^9)\right).$$

The mod 3 result now follows by induction on α, using the fact that

$$\phi(-q) = \phi(-q^9) - 2qf(-q^3,-q^{15}).$$

□

Theorem 3.5. *For $\alpha \geq 1$ and all $n \geq 0$,*

$$\begin{aligned}
ped\left(3^{2\alpha+1}n + \frac{17\times 3^{2\alpha}-1}{8}\right) &\equiv 0 \pmod{2},\\
ped\left(3^{2\alpha+2}n + \frac{11\times 3^{2\alpha+1}-1}{8}\right) &\equiv 0 \pmod{2},\\
ped\left(3^{2\alpha+2}n + \frac{19\times 3^{2\alpha+1}-1}{8}\right) &\equiv 0 \pmod{2},\\
ped\left(3^{2\alpha+1}n + \frac{17\times 3^{2\alpha}-1}{8}\right) &\equiv 0 \pmod{3}, \quad \textit{and}\\
ped\left(3^{2\alpha+2}n + \frac{19\times 3^{2\alpha+1}-1}{8}\right) &\equiv 0 \pmod{3}.
\end{aligned}$$

Proof. The results follow directly from Theorem 3.4, once we observe that $\psi(q)$, $\psi(-q)$ and $\phi(q)$ contain no term in which the power of q is 2 modulo 3, while $\psi(q^3)$ contains no term in which the power of q is 1 or 2 modulo 3. □

Corollary 3.6. *For $\alpha \geq 1$ and all $n \geq 0$,*

$$\begin{aligned}
ped\left(3^{2\alpha+1}n + \frac{17\times 3^{2\alpha}-1}{8}\right) &\equiv 0 \pmod{6} \quad \textit{and}\\
ped\left(3^{2\alpha+2}n + \frac{19\times 3^{2\alpha+1}-1}{8}\right) &\equiv 0 \pmod{6}.
\end{aligned}$$

Proof. These results are immediate from Theorem 3.5. □

We close with a significant result regarding the density of $ped(n)$ modulo 6.

Theorem 3.7. *The function $ped(n)$ is divisible by 6 at least $1/6$ of the time.*

Proof. The arithmetic sequences $9n+7$, $3^{2\alpha+1}n+\frac{17\times 3^{2\alpha}-1}{8}$ and $3^{2\alpha+2}n+\frac{19\times 3^{2\alpha+1}-1}{8}$ (for $\alpha \geq 1$), on which $ped(\cdot)$ is 0 modulo 6, do not intersect. These sequences account for

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ \cdots\ =\frac{1}{6}$$

of all positive integers. □

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