Arithmetic Properties of Partitions with Odd Parts Distinct

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Abstract

In this work, we consider the function $pod(n)$, the number of partitions of an integer n wherein the odd parts are distinct (and the even parts are unrestricted), a function which has arisen in recent work of Alladi. Our goal is to consider this function from an arithmetic point of view in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. We prove a number of results for $pod(n)$ including the following infinite family of congruences: for all $\alpha \geq 0$ and $n \geq 0$,

$$pod\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3.$$

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1 Introduction

We denote by $pod(n)$ the function which enumerates the partitions of n wherein odd parts are distinct (and even parts are unrestricted). This function $pod(n)$ has been considered by many from a product–series point of view as well as from other directions. For example, $pod(n)$ appears in the works of Andrews [2, 3] and Berkovich and Garvan [5]. Moreover, Berkovich and Garvan note that Andrews [4] considered a restricted version of $pod(n)$ wherein each part was required to be larger than 1. In very recent work, Alladi [1] obtained a series expansion for the product generating function for $pod(n)$. However, $pod(n)$ has not previously been studied from an arithmetic viewpoint.

Our goal in this work is to consider $pod(n)$ from an arithmetic point of view in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. In this vein, we will establish the Ramanujan–type congruences: for $\alpha \geq 0$ and $n \geq 0$,

$$pod\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3$$

as well as the 'internal' congruences

$$\begin{aligned} pod(81n+17) &\equiv 5\,pod(9n+2) \pmod{27},\\ pod(81n+44) &\equiv -7\,pod(9n+5) \pmod{27}\\ \text{and}\quad pod(81n+71) &\equiv -pod(9n+8) \pmod{27}. \end{aligned}$$

2 Preliminaries

We use the standard notation

$$(a;q)_\infty = \prod_{n\geq 1}(1-aq^{n-1}),$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots (a_k; q)_\infty.$$

We find that the generating function for $pod(n)$ is, thanks to Jacobi's triple product identity, given by

$$\sum_{n\geq 0} pod(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{(q^2; q^4)_\infty}{(q; q)_\infty} = \frac{1}{(q, q^3, q^4; q^4)_\infty} = \frac{1}{\sum_{-\infty}^{\infty} (-1)^n q^{2n^2-n}} = \frac{1}{\psi(-q)}$$

where

$$\psi(q) = \sum_{-\infty}^{\infty} q^{2n^2-n} = \sum_{n\geq 0} q^{(n^2+n)/2}.$$

We start by noting that

$$\psi(q) = 1 + q + q^3 + q^6 + q^{10} + q^{15} + q^{21} + \cdots = P(q^3) + q\psi(q^9),$$

where

$$P(q) = 1 + q + q^2 + q^5 + q^7 + q^{12} + \cdots$$

and the exponents are the pentagonal numbers $(3n^2 \pm n)/2$. (By way of explanation, the powers in $\psi(q)$ are the triangular numbers $(n^2 + n)/2$, and are either three times a pentagonal number or one more than nine times a triangular number, according as $n \not\equiv 1 \pmod 3$ or $n \equiv 1 \pmod 3$.)

We develop some key machinery in the following lemmas:

Lemma 2.1. *The following identities hold:*

$$P(q^3)^3 + q^3\psi(q^9)^3 = \frac{\psi(q^3)^4}{\psi(q^9)}$$

and

$$P(q)^3 = \frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}$$

Proof. We have, with $\omega = e^{\frac{2\pi i}{3}}$,

$$\begin{aligned}
(q; q)_\infty (\omega q; \omega q)_\infty (\overline{\omega} q; \overline{\omega} q)_\infty &= \prod_{n\geq 1} (1 - q^n)(1 - \omega^n q^n)(1 - \overline{\omega}^n q^n) \\
&= \prod_{3|n} (1 - q^n)^3 \prod_{3\nmid n} (1 - q^{3n}) \\
&= \prod_{n\geq 1} (1 - q^{3n})^4/(1 - q^{9n}) \\
&= \frac{(q^3; q^3)_\infty^4}{(q^9; q^9)_\infty}.
\end{aligned}$$

Again thanks to Jacobi's triple product identity, we have

$$\psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}.$$

So

$$\begin{aligned}
\psi(q)\psi(\omega q)\psi(\overline{\omega}q) &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\cdot\frac{(\overline{\omega}q^2;\overline{\omega}q^2)_\infty^2}{(\omega q;\omega q)_\infty}\cdot\frac{(\omega q^2;\omega q^2)_\infty^2}{(\overline{\omega}q;\overline{\omega}q)_\infty} \\
&= \frac{((q^2;q^2)_\infty(\omega q^2;\omega q^2)_\infty(\overline{\omega}q^2;\overline{\omega}q^2)_\infty)^2}{(q;q)_\infty(\omega q;\omega q)_\infty(\overline{\omega}q;\overline{\omega}q)_\infty} \\
&= \left(\frac{(q^6;q^6)_\infty^4}{(q^{18};q^{18})_\infty}\right)^2\left(\frac{(q^9;q^9)_\infty}{(q^3;q^3)_\infty^4}\right) \\
&= \left(\frac{(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty}\right)^4\left(\frac{(q^9;q^9)_\infty}{(q^{18};q^{18})_\infty^2}\right) \\
&= \frac{\psi(q^3)^4}{\psi(q^9)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
P(q^3)^3+q^3\psi(q^9)^3 &= \left(P(q^3)+q\psi(q^9)\right)\left(P(q^3)+\omega q\psi(q^9)\right)\left(P(q^3)+\overline{\omega}q\psi(q^9)\right) \\
&= \psi(q)\psi(\omega q)\psi(\overline{\omega}q) \\
&= \frac{\psi(q^3)^4}{\psi(q^9)}.
\end{aligned}$$

This proves the first identity. The second identity then follows as an immediate corollary of the first by simple rearrangement and the substitution $q^3 \to q$. □

Lemma 2.2. *The following identity holds:*

$$\frac{1}{\psi(q)} = \frac{\psi(q^9)}{\psi(q^3)^4}(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2)$$

Proof. We have

$$\begin{aligned}
\frac{1}{\psi(q)} &= \frac{1}{P(q^3)+q\psi(q^9)} \\
&= \frac{\left(P(q^3)+\omega q\psi(q^9)\right)\left(P(q^3)+\overline{\omega}q\psi(q^9)\right)}{P(q^3)^3+q^3\psi(q^9)^3} \\
&= \frac{\psi(q^9)}{\psi(q^3)^4}(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2)
\end{aligned}$$

thanks to Lemma 2.1 above. □

3 The Ramanujan–type congeruences

With the above preliminaries in place, we are now in a position to prove the following theorem.

Theorem 3.1. *For $\alpha \geq 0$,*

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+1}n + \frac{5\times 3^{2\alpha+1}+1}{8}\right)q^n &\equiv (-1)^\alpha\frac{\psi(q^3)^3}{\psi(q)^4} \pmod 3, \\
\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n + \frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n &\equiv (-1)^\alpha\frac{\psi(q^3)^4}{\psi(q)^5} \pmod 3.
\end{aligned}$$

Proof. In this proof, all congruences hold to the modulus 3. We have

$$\sum_{n\geq 0}(-1)^n pod(n)q^n = \frac{1}{\psi(q)} = \frac{\psi(q^9)}{\psi(q^3)^4}\left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right).$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, divide by q^2 and replace q^3 by q, we obtain

$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}, \tag{1}$$

which establishes the case $\alpha = 0$ of the first congruence in the Theorem. We now proceed by induction on α.

Suppose

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+1}n + \frac{5\times 3^{2\alpha+1}+1}{8}\right)q^n \equiv (-1)^\alpha \frac{\psi(q^3)^3}{\psi(q)^4} \pmod 3.$$

Then

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+1}n + \frac{5\times 3^{2\alpha+1}+1}{8}\right)q^n \\
&\equiv (-1)^\alpha \psi(q^3)^3 \left(\frac{\psi(q^9)}{\psi(q^3)^4}\left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right)\right)^4 \\
&\equiv (-1)^\alpha \frac{\psi(q^9)^4}{\psi(q^3)^{13}}\left(P(q^3)^8 - qP(q^3)^7\psi(q^9) + q^2P(q^3)^6\psi(q^9)^2 - q^3P(q^3)^5\psi(q^9)^3 + q^4P(q^3)^4\psi(q^9)^4\right. \\
&\qquad \left. -q^5P(q^3)^3\psi(q^9)^5 + q^6P(q^3)^2\psi(q^9)^6 - q^7P(q^3)\psi(q^9)^7 + q^8\psi(q^9)^8\right).
\end{aligned}$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, divide by q^2 and replace q^3 by q, we obtain

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n + \frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n \\
&\equiv (-1)^\alpha \frac{\psi(q^3)^4}{\psi(q)^{13}}\left(P(q)^6\psi(q^3)^2 - qP(q)^3\psi(q^3)^5 + q^2\psi(q^3)^8\right) \\
&\equiv (-1)^\alpha \left\{\frac{\psi(q^3)^6}{\psi(q)^{13}}\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right)^2 - q\frac{\psi(q^3)^9}{\psi(q)^{13}}\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right) + q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}}\right\} \\
&\equiv (-1)^\alpha \left\{\frac{\psi(q^3)^4}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right)^2 - q\frac{\psi(q^3)^8}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right) + q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}}\right\} \\
&\equiv (-1)^\alpha \frac{\psi(q^3)^4}{\psi(q)^5}.
\end{aligned}$$

Now suppose

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n + \frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n \equiv (-1)^\alpha \frac{\psi(q^3)^4}{\psi(q)^5} \pmod 3.$$

Then

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n + \frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n \\
&\equiv (-1)^\alpha \psi(q^3)^4 \left(\frac{\psi(q^9)}{\psi(q^3)^4}\left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right)\right)^5 \\
&\equiv (-1)^\alpha \frac{\psi(q^9)^5}{\psi(q^3)^{16}}\left(P(q^3)^{10} + qP(q^3)^9\psi(q^9) + q^9P(q^3)\psi(q^9)^9 + q^{10}\psi(q^9)^{10}\right).
\end{aligned}$$

If we extract those terms in which the power of q is congruent to 1 modulo 3, divide by $-q$ and replace q^3 by q, we obtain

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+3}n+\frac{5\times 3^{2\alpha+3}+1}{8}\right)q^n\\
&\equiv (-1)^{\alpha+1}\frac{\psi(q^3)^5}{\psi(q)^{16}}\left(P(q)^9\psi(q^3)+q^3\psi(q^3)^{10}\right)\\
&\equiv (-1)^{\alpha+1}\left\{\frac{\psi(q^3)^6}{\psi(q)^{16}}\left(\frac{\psi(q)^4-q\psi(q^3)^4}{\psi(q^3)}\right)^3+q^3\frac{\psi(q^3)^{15}}{\psi(q)^{16}}\right\}\\
&\equiv (-1)^{\alpha+1}\left\{\frac{\psi(q^3)^3}{\psi(q)^{16}}\left(\psi(q)^4-q\psi(q^3)^4\right)^3+q^3\frac{\psi(q^3)^{15}}{\psi(q)^{16}}\right\}\\
&\equiv (-1)^{\alpha+1}\frac{\psi(q^3)^3}{\psi(q)^4}.
\end{aligned}$$

□

We can now prove

Theorem 3.2. *For $\alpha \geq 0$ and $n \geq 0$,*

$$pod\left(3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right)\equiv 0 \pmod 3.$$

Proof. We have

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n+\frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n\equiv (-1)^{\alpha}\frac{\psi(q^3)^4}{\psi(q)^5}\\
&\equiv (-1)^{\alpha}\frac{\psi(q^9)^5}{\psi(q^3)^{16}}\left(P(q^3)^{10}+qP(q^3)^9\psi(q^9)+q^9P(q^3)\psi(q^9)^9+q^{10}\psi(q^9)^{10}\right).
\end{aligned}$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, divide by q^2 and replace q^3 by q, we obtain

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right)q^n\equiv 0 \pmod 3.$$

The result follows. □

Corollary 3.3. *$pod(n)$ is divisible by 3 for at least $\frac{1}{24}$ of all non–negative integers n.*

Proof. The arithmetic sequences $\left\{3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right\}$, $\alpha = 0,\ 1,\ \cdots$ mentioned in Theorem 3.2 do not intersect, and they account for

$$\frac{1}{3^3}+\frac{1}{3^5}+\ \cdots=\frac{1}{24}$$

of all non–negative integers. □

4 The internal congruences

We will now prove

Theorem 4.1.

$$\begin{aligned}
pod(81n+17) &\equiv 5\,pod(9n+2) \pmod{27},\\
pod(81n+44) &\equiv -7\,pod(9n+5) \pmod{27}\\
\textit{and}\quad pod(81n+71) &\equiv -pod(9n+8) \pmod{27}.
\end{aligned}$$

Proof. In this proof, all congruences hold to the modulus 27. Thanks to (1), we have

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4} \\
&= \psi(q^3)^3\left(\frac{\psi(q^9)}{\psi(q^3)^4}\left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right)\right)^4 \quad \text{from Lemma 2.2} \\
&\equiv \frac{\psi(q^9)^4}{\psi(q)^{13}}\left(P(q^3)^8 - 4qP(q^3)^7\psi(q^9) + 10q^2P(q^3)^6\psi(q^9)^2 + 11q^3P(q^3)^5\psi(q^9)^3 - 8q^4P(q^3)^4\psi(q^9)^4\right. \\
&\qquad \left. +11q^5P(q^3)^3\psi(q^9)^5 + 10q^6P(q^3)^2\psi(q^9)^6 - 4q^7P(q^3)\psi(q^9)^7 + q^8\psi(q^9)^8\right) \qquad (2)
\end{aligned}$$

thanks to the Binomial Theorem followed by reduction of all coefficients modulo 27. We now extract those terms in (2) in which the power of q is congruent to 0 modulo 3, replace q^3 by q throughout, replace $P(q)^3$ via Lemma 2.1, and again apply the Binomial Theorem to obtain

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+2)q^n \\
&\equiv P(q)^2\left\{\frac{\psi(q^3)^4}{\psi(q)^{13}}\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right)^2 + 11q\frac{\psi(q^3)^7}{\psi(q)^{13}}\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right) + 10q^2\frac{\psi(q^3)^{10}}{\psi(q)^{13}}\right\} \\
&\equiv P(q)^2\left\{\frac{\psi(q^3)^2}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right)^2 + 11q\frac{\psi(q^3)^6}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right) + 10q^2\frac{\psi(q^3)^{10}}{\psi(q)^{13}}\right\} \\
&\equiv P(q)^2\left\{\frac{\psi(q^3)^2}{\psi(q)^5} + 9q\frac{\psi(q^3)^6}{\psi(q)^9}\right\}. \qquad (3)
\end{aligned}$$

Similar work, beginning with (2), extracting those terms in which the power of q is congruent to 1 modulo 3, dividing by $-q$ and replacing q^3 by q throughout, yields

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+5)q^n \\
&\equiv P(q)\left\{4\frac{\psi(q^3)^3}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right)^2 + 8q\frac{\psi(q^3)^7}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right) + 4q^2\frac{\psi(q^3)^{11}}{\psi(q)^{13}}\right\} \\
&\equiv 4P(q)\frac{\psi(q^3)^3}{\psi(q)^5}. \qquad (4)
\end{aligned}$$

Lastly, analogous work gives

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \\
&\equiv 10\frac{\psi(q^3)^4}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right)^2 + 11q\frac{\psi(q^3)^8}{\psi(q)^{13}}\left(\psi(q)^4 - q\psi(q^3)^4\right) + q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}} \\
&\equiv 10\frac{\psi(q^3)^4}{\psi(q)^5} - 9q\frac{\psi(q^3)^8}{\psi(q)^9}. \qquad (5)
\end{aligned}$$

We now use Lemma 2.2 in (5), along with similar applications of the Binomial Theorem and the reduction of all coefficients modulo 27, to obtain

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+8)q^n\\
&\equiv 10\frac{\psi(q^9)^5}{\psi(q^3)^{16}}\big(P(q^3)^{10}-5qP(q^3)^9\psi(q^9)-12q^2P(q^3)^8\psi(q^9)^2-3q^3P(q^3)^7\psi(q^9)^3\\
&\qquad -9q^4P(q^3)^6\psi(q^9)^4+3q^5P(q^3)^5\psi(q^9)^5-9q^6P(q^3)^4\psi(q^9)^6-3q^7P(q^3)^3\psi(q^9)^7\\
&\qquad -12q^8P(q^3)^2\psi(q^9)^8-5q^9P(q^3)\psi(q^9)^9+q^{10}\psi(q^9)^{10}\big)\\
&-9q\frac{\psi(q^9)^9}{\psi(q^3)^{28}}\big(P(q^3)^{18}-9qP(q^3)^{17}\psi(q^9)-9q^2P(q^3)^{16}\psi(q^9)^2+6q^3P(q^3)^{15}\psi(q^9)^3\\
&\qquad +9q^4P(q^3)^{14}\psi(q^9)^4+9q^5P(q^3)^{13}\psi(q^9)^5-12q^6P(q^3)^{12}\psi(q^9)^6-9q^7P(q^3)^{11}\psi(q^9)^7\\
&\qquad -9q^8P(q^3)^{10}\psi(q^9)^8-7q^9P(q^3)^9\psi(q^9)^9-9q^{10}P(q^3)^8\psi(q^9)^{10}-9q^{11}P(q^3)^7\psi(q^9)^{11}\\
&\qquad -12q^{12}P(q^3)^6\psi(q^9)^{12}+9q^{13}P(q^3)^5\psi(q^9)^{13}+9q^{14}P(q^3)^4\psi(q^9)^{14}+6q^{15}P(q^3)^3\psi(q^9)^{15}\\
&\qquad -9q^{16}P(q^3)^2\psi(q^9)^{16}-9q^{17}P(q^3)\psi(q^9)^{17}+q^{18}\psi(q^9)^{18}\big).
\end{aligned}$$

From here we extract those terms in which the powers of q are congruent to 1 modulo 3, divide by $-q$ and replace q^3 by q, use Lemma 2.2 once again along with the Binomial Theorem to obtain

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(27n+17)q^n\\
&\equiv -4\frac{\psi(q^3)^3}{\psi(q)^{16}}\left(\psi(q)^4-q\psi(q^3)^4\right)^3+9q\frac{\psi(q^3)^7}{\psi(q)^{16}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2+3q^2\frac{\psi(q^3)^{11}}{\psi(q)^{16}}\left(\psi(q)^4-q\psi(q^3)^4\right)\\
&\quad -10q^3\frac{\psi(q^3)^{15}}{\psi(q)^{16}}+9\frac{\psi(q^3)^3}{\psi(q)^{28}}\left(\psi(q)^4-q\psi(q^3)^4\right)^6-9q^3\frac{\psi(q^3)^{15}}{\psi(q)^{28}}\left(\psi(q)^4-q\psi(q^3)^4\right)^3+9q^6\frac{\psi(q^3)^{27}}{\psi(q)^{28}}\\
&\equiv 5\frac{\psi(q^3)^3}{\psi(q)^4}-6q\frac{\psi(q^3)^7}{\psi(q)^8}\\
&\equiv 5\frac{\psi(q^9)^4}{\psi(q^3)^{13}}\big(P(q^3)^8-4qP(q^3)^7\psi(q^9)+10q^2P(q^3)^6\psi(q^9)^2+11q^3P(q^3)^5\psi(q^9)^3-8q^4P(q^3)^4\psi(q^9)^4\\
&\qquad +11q^5P(q^3)^3\psi(q^9)^5+10q^6P(q^3)^2\psi(q^9)^6-4q^7P(q^3)\psi(q^9)^7+q^8\psi(q^9)^8\big)\\
&-6q\frac{\psi(q^9)^8}{\psi(q^3)^{25}}\big(P(q^3)^{16}-8qP(q^3)^{15}\psi(q^9)+9q^2P(q^3)^{14}\psi(q^9)^2-4q^3P(q^3)^{13}\psi(q^9)^3-4q^4P(q^3)^{12}\psi(q^9)^4\\
&\qquad +9q^5P(q^3)^{11}\psi(q^9)^5+q^6P(q^3)^{10}\psi(q^9)^6+10q^7P(q^3)^9\psi(q^9)^7+10q^9P(q^3)^7\psi(q^9)^9\\
&\qquad +q^{10}P(q^3)^6\psi(q^9)^{10}+9q^{11}P(q^3)^5\psi(q^9)^{11}-4q^{12}P(q^3)^4\psi(q^9)^{12}-4q^{13}P(q^3)^3\psi(q^9)^{13}\\
&\qquad +9q^{14}P(q^3)^2\psi(q^9)^{14}-8q^{15}P(q^3)\psi(q^9)^{15}+q^{16}\psi(q^9)^{16}\big).
\end{aligned}$$

We now apply similar techniques to those above to finally obtain the congruences in this theorem.

First,

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(81n+17)\\
&\equiv P(q)^2\left\{5\frac{\psi(q^3)^2}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2+q\frac{\psi(q^3)^6}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)-4q^2\frac{\psi(q^3)^{10}}{\psi(q)^{13}}\right\}\\
&\equiv P(q)^2\left\{5\frac{\psi(q^3)^2}{\psi(q)^5}-9q\frac{\psi(q^3)^6}{\psi(q)^9}\right\}\\
&\equiv 5\sum_{n\geq 0}(-1)^n pod(9n+2)q^n \quad \text{thanks to (3),}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(81n+44)q^n\\
&\equiv P(q)\left\{-7\frac{\psi(q^3)^3}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2+13q\frac{\psi(q^3)^7}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)-7q^2\frac{\psi(q^3)^{11}}{\psi(q)^{13}}\right.\\
&+6\frac{\psi(q^3)^3}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^5+3q\frac{\psi(q^3)^7}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^4+6q^2\frac{\psi(q^3)^{11}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^3\\
&\left.+6q^3\frac{\psi(q^3)^{15}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2+3q^4\frac{\psi(q^3)^{19}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)+6q^5\frac{\psi(q^3)^{23}}{\psi(q)^{25}}\right\}\\
&\equiv -P(q)\frac{\psi(q^3)^3}{\psi(q)^5}\\
&\equiv -7\sum_{n\geq 0}(-1)^n pod(9n+5)q^n \quad \text{via (4)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(81n+71)q^n\\
&\equiv -4\frac{\psi(q^3)^4}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2+q\frac{\psi(q^3)^8}{\psi(q)^{13}}\left(\psi(q)^4-q\psi(q^3)^4\right)+5q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}}\\
&-6\frac{\psi(q^3)^4}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^5-3q\frac{\psi(q^3)^8}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^4-6q^2\frac{\psi(q^3)^{12}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^3\\
&-6q^3\frac{\psi(q^3)^{16}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)^2-3q^4\frac{\psi(q^3)^{20}}{\psi(q)^{25}}\left(\psi(q)^4-q\psi(q^3)^4\right)-6q^5\frac{\psi(q^3)^{24}}{\psi(q)^{25}}\\
&\equiv -10\frac{\psi(q^3)^4}{\psi(q)^5}+9q\frac{\psi(q^3)^8}{\psi(q)^9}\\
&\equiv -\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \quad \text{from (5).}
\end{aligned}$$

□

5 Concluding remarks

We know rather more about *pod*(n); for instance, the result in Theorem 3.2 is best possible, in the sense that the modulus, 3, cannot be replaced by a higher power of 3:

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right)q^n \equiv 3(-1)^{\alpha+1}P(q)^2\frac{\psi(q^3)^5}{\psi(q)^8} \pmod 9.$$

Indeed, for example,

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(27n+26)q^n \\
= \;& P(q)^2\left\{501\frac{\psi(q^3)^5}{\psi(q)^8} - 29484q\frac{\psi(q^3)^9}{\psi(q)^{12}} + 574533q^2\frac{\psi(q^3)^{13}}{\psi(q)^{16}} - 5130702q^3\frac{\psi(q^3)^{17}}{\psi(q)^{20}} + 23422770q^4\frac{\psi(q^3)^{21}}{\psi(q)^{24}}\right. \\
&\left. -57631824q^5\frac{\psi(q^3)^{25}}{\psi(q)^{28}} + 77058945q^6\frac{\psi(q^3)^{29}}{\psi(q)^{32}} - 52612659q^7\frac{\psi(q^3)^{33}}{\psi(q)^{36}} + 14348907q^8\frac{\psi(q^3)^{37}}{\psi(q)^{40}}\right\} \\
= \;& -3P(q)^2\frac{\psi(q^3)^5}{\psi(q)^8} + 9P(q)^2\left\{56\frac{\psi(q^3)^5}{\psi(q)^8} - 3828q\frac{\psi(q^3)^9}{\psi(q)^{12}} + 63837q^2\frac{\psi(q^3)^{13}}{\psi(q)^{16}} - 570078q^3\frac{\psi(q^3)^{17}}{\psi(q)^{20}}\right. \\
&+2602530q^4\frac{\psi(q^3)^{21}}{\psi(q)^{24}} - 6403536q^5\frac{\psi(q^3)^{25}}{\psi(q)^{28}} + 8562105q^6\frac{\psi(q^3)^{29}}{\psi(q)^{32}} \\
&\left. -5845851q^7\frac{\psi(q^3)^{33}}{\psi(q)^{36}} + 1594323q^8\frac{\psi(q^3)^{37}}{\psi(q)^{40}}\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n pod(9n+2)q^n &= P(q)^2\left\{\frac{\psi(q^3)^2}{\psi(q)^5} - 18q\frac{\psi(q^3)^6}{\psi(q)^9} + 27q^2\frac{\psi(q^3)^{10}}{\psi(q)^{13}}\right\}, \\
\sum_{n\geq 0}(-1)^n pod(9n+5)q^n &= P(q)\left\{4\frac{\psi(q^3)^3}{\psi(q)^5} - 27q\frac{\psi(q^3)^7}{\psi(q)^9} + 27q^2\frac{\psi(q^3)^{11}}{\psi(q)^{13}}\right\}, \\
\sum_{n\geq 0}(-1)^n pod(9n+8)q^n &= 10\frac{\psi(q^3)^4}{\psi(q)^5} - 36q\frac{\psi(q^3)^8}{\psi(q)^9} + 27q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}},
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n pod(81n+17)q^n &\equiv P(q)^2\left\{5\frac{\psi(q^3)^2}{\psi(q)^5} + 18q\frac{\psi(q^3)^6}{\psi(q)^9} - 27q^2\frac{\psi(q^3)^{10}}{\psi(q)^{13}}\right\} \pmod{81}, \\
\sum_{n\geq 0}(-1)^n pod(81n+44)q^n &\equiv P(q)\left\{-\frac{\psi(q^3)^3}{\psi(q)^5} - 27q\frac{\psi(q^3)^7}{\psi(q)^9} - 27q^2\frac{\psi(q^3)^{11}}{\psi(q)^{13}}\right\} \pmod{81}, \\
\sum_{n\geq 0}(-1)^n pod(81n+71)q^n &\equiv 17\frac{\psi(q^3)^4}{\psi(q)^5} + 9q\frac{\psi(q^3)^8}{\psi(q)^9} - 27q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}} \pmod{81},
\end{aligned}$$

from which it follows that the modulus, 27, in Theorem 4.1 cannot be replaced by 81.

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