PARTITIONS WITH PARTS IN A FINITE SET

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ABSTRACT. For a finite set A of positive integers, we study the partition function $p_A(n)$. This function enumerates the partitions of the positive integer n into parts in A. We give simple proofs of some known and unknown identities and congruences for $p_A(n)$. For n in a special residue class, $p_A(n)$ is a polynomial in n. We examine these polynomials for linear factors, and the results are applied to a restricted m-ary partition function. We extend the domain of p_A and prove a reciprocity formula with supplement. In closing we consider an asymptotic formula for $p_A(n)$ and its refinement.

1. INTRODUCTION

Let A be a non-empty set of natural numbers. An (unordered) partition of a natural number n with parts in A is a sequence $p_1, p_2, \ldots, p_r$ of, not necessarily distinct, elements p_i in A, such that

$$n = p_1 + p_2 + \cdots + p_r. \tag{1.1}$$

The order of the parts p_i does not matter. Therefore one often chooses to consider partitions with decreasing (or increasing) parts only.

Let $p_A(n)$ denote the number of partitions (1.1) of n with $p_i \in A$. Putting $p_A(0) = 1$, we can write the generating function $F(x) = \sum_{n=0}^{\infty} p_A(n)x^n$ as

$$F(x) = \prod_{a \in A} \frac{1}{1 - x^a}.$$

In particular, if $A = \mathbb{N}$, the set of natural numbers, then $p_A(n) = p(n)$, the number of unrestricted partitions of n, and the result

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

was published by Euler in 1748. There is an abundance of literature on the partition function $p(n)$. Among the main issues studied are divisibility properties and asymptotics.

In this paper we consider the case where the set A is *finite*. Let A consist of the positive integers $a_0, a_1, \ldots, a_{k-1}$. Then $p_A(n)$ is equal to the number

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of solutions $(x_0, x_1, \ldots, x_{k-1})$ in non-negative integers x_i of the equation

$$n = a_0x_0 + a_1x_1 + \cdots + a_{k-1}x_{k-1}. \tag{1.2}$$

From this point of view, $p_A(n)$ is defined (and the results below hold) even if the a_i are not all distinct, that is, if A is a finite *multiset* of natural numbers. We assume that the numbers in A are relatively prime. This does not imply any loss of generality.

This paper is organized as follows. In Section 2 we show that $p_A(n)$ is a quasi-polynomial in n of degree $k-1$; that is, for n in a fixed residue class modulo a certain number, $p_A(n)$ is a polynomial in n with coefficients in $\mathbb{Q}$. In Section 3 we show that these polynomials may have (several) integer zeros. In Section 4 we construct a class of such polynomials with a non-integral rational zero, and we also construct a class of polynomials with a double (integer) zero. In Section 5 we apply some of the previous results to a special choice of the set A. In Section 6 we discuss some of the previous material from another point of view, while we extend the domain of p_A to all of $\mathbb{Z}$ and prove a reciprocity formula with supplement. Finally, in Section 7 we close with a simple arithmetic proof of a well-known asymptotic result for $p_A(n)$. We also add a remark on the error term when using the approximation of $p_A(n)$ coming from the pole $x = 1$ of $F(x)$.

2. Finite A

If $k = 1$, then $a_0 = 1$ and $p_A(n) = 1$ for all $n \geq 0$. Also if $k = 2$ the situation is simple. Any non-negative integer can uniquely be written as $a_0a_1n + a_0r + a_1s$ with $n \geq -1$, $0 \leq r < a_1$, $0 \leq s < a_0$. Then clearly,

$$p_A(a_0a_1n + a_0r + a_1s) = n + 1.$$

Now to general k. Let α be a positive common multiple of $a_0, a_1, \ldots, a_{k-1}$. Then

$$F(x) = \prod_{i=0}^{k-1} \frac{1}{1 - x^{a_i}} = \frac{f(x)}{(1 - x^\alpha)^k}, \tag{2.1}$$

where

$$f(x) = \prod_{i=0}^{k-1} \frac{1 - x^\alpha}{1 - x^{a_i}} = \prod_{i=0}^{k-1} \sum_{j=0}^{\alpha_i - 1} x^{ja_i}, \tag{2.2}$$

and where $\alpha_i = \alpha/a_i$. Thus there are non-negative integers f_i such that

$$f(x) = f_0 + f_1x + \cdots + f_dx^d,$$

where $d = \alpha k - \sigma$ for $\sigma = a_0 + a_1 + \cdots + a_{k-1}$. Note that the polynomial $f(x)$ is reciprocal, that is,

$$x^d f\left(\frac{1}{x}\right) = f(x),$$

so that $f_{d-i} = f_i$ for $i = 0, 1, \ldots, d$.

Recall the binomial series

$$\frac{1}{(1-y)^k} = \sum_{n=0}^{\infty}(-1)^n\binom{-k}{n}y^n = \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}y^n.$$

Putting $y = x^\alpha$ in this result, we have, by (2.1),

$$\sum_{n=0}^{\infty}p_A(n)x^n = \sum_{i=0}^{d}f_i x^i \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^{\alpha n}.$$

Let r be an integer in the interval $0 \le r < \alpha$. We extract the terms where the exponent of x is congruent to r mod α to get

$$\sum_{n=0}^{\infty}p_A(\alpha n+r)x^{\alpha n+r} = \sum_{i\ge 0}f_{\alpha i+r}x^{\alpha i+r}\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}x^{\alpha j},$$

and where $f_j = 0$ if $j > d$. We cancel x^r and replace x^α by x. This gives us

$$\begin{aligned}\sum_{n=0}^{\infty}p_A(\alpha n+r)x^n &= \sum_{i\ge 0}f_{\alpha i+r}\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}x^{i+j}\\ &= \sum_{n=0}^{\infty}\sum_{i\ge 0}f_{\alpha i+r}\binom{n-i+k-1}{k-1}x^n.\end{aligned}$$

If $f_{\alpha i+r} \ne 0$, then $\alpha i + r \le d$, which holds if and only if $i \le k - \lceil (r+\sigma)/\alpha \rceil$. Thus we have (2.3) below. Since this result holds for all $n \ge 0$, the coefficients $f_{\alpha i+r}$ are unique.

Theorem 1. *There are unique integers $f_{\alpha i+r}$ such that for all $n \ge 0$,*

$$p_A(\alpha n+r) = \sum_{i=0}^{k-\kappa}f_{\alpha i+r}\binom{n-i+k-1}{k-1}, \tag{2.3}$$

where $\kappa = \lceil (r+\sigma)/\alpha \rceil$.

For each $r = 0, 1 \ldots, \alpha-1$, we now have that $p_A(\alpha n + r)$ is a polynomial of degree at most $k-1$ in n with rational coefficients. It follows that there exist rational numbers $c_i = c_i(r)$ such that

$$p_A(n) = c_{k-1}n^{k-1} + c_{k-2}n^{k-2} + \cdots + c_0. \tag{2.4}$$

Since the coefficients depend upon the residue class of n mod α, this will usually not be a polynomial in n. An expression of this type is called a *quasi-polynomial* (of quasi-period α), cf. [16, p. 210]. The result (2.4) goes back at least to Bell [4], who used partial fraction decomposition of the generating function to prove it.

3. Congruences

We define the rising factorial (the "Pochhammer symbol") by

$$\langle u\rangle_i = u(u+1)\cdots(u+i-1) \qquad \text{for } i \geq 1.$$

We also put $\langle u\rangle_0 = 1$. Here u is not necessarily an integer.

Integers denoted r and s will be connected by the relation

$$r + s + \sigma \equiv 0 \pmod{\alpha}, \qquad 0 \leq r, s < \alpha. \tag{3.1}$$

Writing $r + s + \sigma = \alpha\kappa$, we have

$$\kappa = \left\lceil \frac{r+\sigma}{\alpha} \right\rceil = \left\lceil \frac{s+\sigma}{\alpha} \right\rceil = \frac{r+s+\sigma}{\alpha}.$$

Due to the symmetry in r and s, to each formula below containing r and s, we can obtain a corresponding dual formula by interchanging r and s.

Multiplying (2.3) through by $(k-1)!$ and rearranging the factorials in the "numerators" of the the binomial coefficients, we get

$$(k-1)!\, p_A(\alpha n + r) = \langle n+1\rangle_{\kappa-1} \sum_{i=0}^{k-\kappa} f_{\alpha i+r} P_i(n), \tag{3.2}$$

where

$$P_i(n) = (-1)^i \langle -n\rangle_i \langle n+\kappa\rangle_{k-\kappa-i}.$$

Let $N \geq 2$ be an integer. By (3.2), we get

$$(k-1)!\, p_A\left(\alpha\left((k-1)!\, N + n\right) + r\right) \equiv (k-1)!\, p_A(\alpha n + r) \pmod{(k-1)!\, N}.$$

Cancelling $(k-1)!$ and writing n for $\alpha n + r$, we have

$$p_A\left((k-1)!\, \alpha N + n\right) \equiv p_A(n) \pmod{N}.$$

Thus the sequence $\{p_A(n)\}_{n\geq 0}$ is periodic mod N with period $(k-1)!\,\alpha N$. Similarly, if N is prime to $(k-1)!$, then αN is a period. More generally, the sequence $\{p_A(n)\}_{n\geq 0}$ is periodic mod $N/\gcd((k-1)!, N)$ with period αN.

By (3.2), we have the following congruence.

Theorem 2. *For all $n \geq 0$,*

$$(k-1)!\, p_A(\alpha n + r) \equiv 0 \pmod{\langle n+1\rangle_{\kappa-1}}.$$

In particular, if $\max\{0, \alpha + 1 - \sigma\} \leq r < \alpha$, then

$$(k-1)!\, p_A(\alpha n + r) \equiv 0 \pmod{n+1} \tag{3.3}$$

for all $n \geq 0$. By replacing n by $n-1$, this result may equivalently be given the following form. Suppose that $0 < t \leq \min\{\alpha, \sigma - 1\}$. Then

$$(k-1)!\, p_A(\alpha n - t) \equiv 0 \pmod{n} \tag{3.4}$$

for all $n \geq 1$.

Substituting $(k-1)!\, n$ for n in (3.3) and (3.4), we get

$$p_A((k-1)!\, \alpha n + r) \equiv 0 \pmod{(k-1)!\, n + 1}$$

if $\max\{0, \alpha + 1 - \sigma\} \le r < \alpha$, and

$$p_A((k-1)!\,\alpha n - t) \equiv 0 \pmod{n}$$

if $0 < t \le \min\{\alpha, \sigma - 1\}$.

These congruences are not of "Ramanujan type". But they do imply results with a more traditional appearance. For example, if ℓ is a prime, $\ell \ge k$, then (3.4) gives

$$p_A(\ell\alpha n - t) \equiv 0 \pmod{\ell}$$

if $0 < t \le \min\{\alpha, \sigma - 1\}$.

Let us present another result of this type. Set $a_0 = \ell^q \ge k$, where ℓ is a prime, q a positive integer, and $k \equiv \ell \pmod 2$. Put $\lambda = \operatorname{lcm}\{a_1, a_2, \ldots, a_{k-1}\}$ and $\alpha = \lambda\ell^q$. Let u and v be connected by the relation

$$u + v + \sigma \equiv 0 \pmod{\ell^q}, \qquad 0 \le u, v < \ell^q.$$

By using an idea of Kronholm [6], we will show that the sequence

$$\{p_A(\ell^q n + u) + p_A(\ell^q n + v)\}_{n \ge 0} \tag{3.5}$$

is periodic mod ℓ with period λ, and in particular,

$$p_A(\alpha n - \ell^q t + u) + p_A(\alpha n - \ell^q t + v) \equiv 0 \pmod{\ell} \tag{3.6}$$

if $0 < t < (u + v + \sigma)/\ell^q$.

Let us look at the proof. Set

$$K(x) = \frac{(1 - x^\lambda)^{\ell^q}}{(1 - x^{a_1}) \cdots (1 - x^{a_{k-1}})}$$

Putting $\gamma = \deg K(x)$, we have $\gamma = (\lambda + 1)\ell^q - \sigma$. Thus we may write $K(x) = k_0 + k_1 x + \cdots + k_\gamma x^\gamma \in \mathbb{Z}[x]$. Since $\ell \equiv k \pmod 2$, we have

$$x^\gamma K\,(1/x) = -K(x),$$

that is, $K(x)$ is *anti-reciprocal*. Equivalently, we have $k_{\gamma - i} = -k_i$ for $i = 0, 1, \ldots, \gamma$.

Let $\gamma^* = \lambda + 1 - (u + v + \sigma)/\ell^q$. Then $0 \le \gamma^* \le \lambda$. We have $k_{\ell^q i + u} = -k_{\ell^q(\gamma^* - i) + v}$. Putting

$$K_u(x) = \sum_{i \ge 0} k_{\ell^q i + u} x^i = \sum_{i=0}^{\gamma^*} k_{\ell^q i + u} x^i,$$

it follows that

$$K_u(x) = -x^{\gamma^*} K_v\,(1/x)\,.$$

For

$$K^*(x) = K_u(x) + K_v(x)$$

we then have

$$x^{\gamma^*} K^*\,(1/x) = -K^*(x).$$

Setting $x = 1$, we see that $K^*(1) = 0$, so we may write

$$\frac{K^*(x)}{1-x} = \sum_{i=0}^{\gamma^*-1} g_i x^i \in \mathbb{Z}[x],$$

where, as usual, an empty sum is taken as zero. Next,

$$\sum_{n=0}^{\infty} p_A(n)x^n = \frac{K(x)}{(1-x^{\ell q})(1-x^{\lambda})^{\ell q}} \equiv \frac{K(x)}{(1-x^{\ell q})(1-x^{\lambda \ell q})} \pmod{\ell}.$$

Comparing terms where the exponent of x is congruent u mod ℓ^q, cancelling x^u, and replacing $x^{\ell q}$ by x, gives

$$\sum_{n=0}^{\infty} p_A(\ell^q n + u)x^n \equiv \frac{K_u(x)}{(1-x)(1-x^{\lambda})} \pmod{\ell}.$$

It follows that

$$\sum_{n=0}^{\infty}(p_A(\ell^q n + u) + p_A(\ell^q n + v))x^n \equiv \frac{K^*(x)}{(1-x)(1-x^{\lambda})} \equiv \sum_{i=0}^{\gamma^*-1} g_i x^i \frac{1}{1-x^{\lambda}} \pmod{\ell}.$$

For $0 \le w < \lambda$, we now have

$$\sum_{n=0}^{\infty} \left(p_A(\ell^q(\lambda n + w) + u) + p_A(\ell^q(\lambda n + w) + v)\right) x^n \equiv g_w \sum_{n=0}^{\infty} x^n \pmod{\ell},$$

that is,

$$p_A(\ell^q(\lambda n + w) + u) + p_A(\ell^q(\lambda n + w) + v) \equiv g_w \pmod{\ell},$$

where $g_w = 0$ if $w \ge \gamma^*$. Now we see that the sequence (3.5) is periodic mod ℓ with period λ. Moreover, if $\gamma^* \le w < \lambda$, then

$$p_A(\alpha n + \ell^q w + u) + p_A(\alpha n + \ell^q w + v) \equiv 0 \pmod{\ell}.$$

Replacing w by $\lambda - t$ and writing $n - 1$ for n, we obtain (3.6) under the given condition.

Finally, taking $A = \{1, 2, \ldots, \ell\}$, ℓ odd, we have $\sigma = \ell(\ell+1)/2$, so we can set $u = v = 0$. By (3.5) the sequence $\{2p_A(\ell n)\}_{n\ge 0}$, that is, the sequence $\{p_A(\ell n)\}_{n\ge 0}$, is periodic mod ℓ with period λ. Moreover, since ℓ is odd, (3.6) gives

$$p_A(\alpha n - \ell t) \equiv 0 \pmod{\ell}$$

if $1 \le t \le (\ell-1)/2$. These results for $A = \{1, 2, \ldots, \ell\}$ with ℓ an odd prime are the results recently obtained by Kronholm [6].

4. More Linear Factors

Since the polynomial $f(x)$ is reciprocal, we have

$$f_{\alpha(k-\kappa-i)+r} = f_{\alpha i+s},$$

and by reversing the summation order in (2.3), we get

$$p_A(\alpha n + r) = \sum_{i=0}^{k-\kappa} f_{\alpha i+s} \binom{n+\kappa+i-1}{k-1}, \tag{4.1}$$

while (3.2) becomes

$$(k-1)!\, p_A(\alpha n + r) = \langle n+1\rangle_{\kappa-1} \sum_{i=0}^{k-\kappa} f_{\alpha i+s} Q_i(n), \tag{4.2}$$

where

$$Q_i(n) = P_{k-\kappa-i}(n) = (-1)^{k-\kappa-i}\langle -n\rangle_{k-\kappa-i}\langle n+\kappa\rangle_i = (-1)^{k-\kappa} P_i(-n-\kappa).$$

Various results can now be deduced by combining (3.2) with (4.2) and its dual.

Let us look at an example. Assuming $k \not\equiv \kappa \pmod 2$, we have $P_i(-\kappa/2) + Q_i(-\kappa/2) = 0$. Suppose that $r = s$. By adding (3.2) and (4.2), we then see that $p_A(\alpha n + r)$ is divisible by $(n+\kappa/2)\langle n+1\rangle_{\kappa-1}$ in $\mathbb{Q}[n]$. If k is even, then κ is odd, and the polynomial $(k-1)!\, p_A(\alpha n + r)$ in $\mathbb{Z}[n]$ is divisible by $n+\kappa/2$ in $\mathbb{Q}[n]$. By Gauss' lemma for polynomials, $(k-1)!\, p_A(\alpha n + r)$ is then divisible by the primitive polynomial $2n+\kappa$ in $\mathbb{Z}[n]$. Thus we have

$$(k-1)!\, p_A(\alpha n + r) \equiv 0 \pmod{(2n+\kappa)\langle n+1\rangle_{\kappa-1}}.$$

For k odd and κ even, we do not get the bonus factor 2, and we have

$$(k-1)!\, p_A(\alpha n + r) \equiv 0 \pmod{(n+\kappa/2)\langle n+1\rangle_{\kappa-1}}. \tag{4.3}$$

Notice, however, that this modulus contains the factor $(n+\kappa/2)^2$.

5. A Special Case: m-ary Partitions

Let $m \geq 2$ be an integer. In this section we set

$$a_i = m^i \qquad \text{for } i = 0, 1, \ldots, k-1.$$

In this case, let us write $p_A(n) = b_{m,k}(n)$. This restricted m-ary partition function $b_{m,k}(n)$ enumerates the representations of n of the form

$$n = m^{\varepsilon_0} + m^{\varepsilon_1} + \cdots + m^{\varepsilon_j},$$

with $\varepsilon_i \in \mathbb{Z}$ and $0 \leq \varepsilon_0 \leq \varepsilon_1 \leq \ldots \leq \varepsilon_j < k$. We also have that $b_{m,k}(n)$ is equal to the number of representations of n on the form

$$n = \delta_0 + \delta_1 m + \delta_2 m^2 + \cdots,$$

where $\delta_i \in \mathbb{Z}$ and $0 \leq \delta_i < m^k$.

We set $\alpha = m^{k-1}$. For $k \geq 2$, we then have, by Theorem 1,

$$b_{m,k}(m^{k-1}n + r) = \sum_{i=0}^{k-2} f_{m^{k-1}i+r} \binom{n-i+k-1}{k-1}$$

for unique integers f_j. For $m = 2$, this is essentially Theorem 3.6 in Reznick [12].

In a series of papers (see [13, 14] and the references therein) it has been shown that $b_{m,k}(n)$ possesses certain divisibility properties. From Section 3 we now get divisibility properties of a rather different type. By (3.3), we have

$$(k-1)!\, b_{m,k}(n) \equiv 0 \pmod{\lfloor n/m^{k-1} \rfloor + 1}. \tag{5.1}$$

Moreover, by Theorem 2,

$$(k-1)!\, b_{m,k}(m^{k-1}n + r) \equiv 0 \pmod{(n+1)(n+2)} \tag{5.2}$$

if

$$m^{k-1} - \frac{m^{k-1}-1}{m-1} < r < m^{k-1}. \tag{5.3}$$

If $m = 2$, then (5.1) and (5.2) hold (under the given conditions) with the factor $(k-1)!$ on each of the left hand sides replaced by ω_{k-1}, the *odd* part of $(k-1)!$. We also have for *odd* $k \geq 3$,

$$\omega_{k-1} b_{2,k}(2^{k-1}n) \equiv 0 \pmod{(n+1)^2}, \tag{5.4}$$

and for *even* $k \geq 4$,

$$\omega_{k-1} b_{2,k}(2^{k-1}n) \equiv 0 \pmod{(n+1)(2n+1)(2n+3)}.$$

Proofs of these results are given in [15].

Let us here and now just prove that for odd $k \geq 3$,

$$(k-1)!\, b_{2,k}(2^{k-1}n) \equiv 0 \pmod{(n+1)^2}, \tag{5.5}$$

which is a slightly weaker version of (5.4). If we take $A = \{1, 2, 2^2, \ldots, 2^{k-1}\}$, then α is even while σ is odd, and the results of Section 4 are not directly applicable. However, a bisection of the generating function of $b_{2,k}(n)$ gives

$$\sum_{n=0}^{\infty} b_{2,k}(2n)x^n = \frac{1}{1-x} \prod_{i=0}^{k-2} \frac{1}{1-x^{2^i}}.$$

Now, take A as the multiset $A = \{1, 1, 2, 2^2, \ldots, 2^{k-2}\}$, and put $\alpha = 2^{k-2}$. Then $\sigma = 2^{k-1} = 2\alpha$, so we can take $r = s = 0$. We have $\kappa = 2$ and (5.5) follows from (4.3).

6. Linear Recurrence

In this section we look at some of the previous material from another viewpoint. Set

$$Q(x) = \prod_{a \in A} (1 - x^a),$$

so that

$$F(x) = \sum_{n=0}^{\infty} p_A(n) x^n = \frac{1}{Q(x)}.$$

Every zero γ of $Q(x)$ satisfies $\gamma^\alpha = 1$. The zero $x = 1$ has multiplicity k, and all the other zeros of $Q(x)$ have lower multiplicity. Hence, $p_A(n)$ is a quasi-polynomial in n of quasi-period α and of degree $k - 1$; cf. [16, Proposition 4.4.1]. Thus (2.4) holds and $p_A(\alpha n + r)$ is a polynomial in n of degree $k - 1$. The abelian group of all polynomials in n of degree at most $k - 1$ with complex coefficients and which map non-negative integers to non-negative integers is free with basis

$$\left\{ \binom{n - i + k - 1}{k - 1} \mid i = 0, 1, \ldots, k - 1 \right\};$$

cf. [16, p. 209]. Thus there are unique integers f_j such that (2.3) holds.

Next, expand $Q(x)$ to get

$$Q(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_\sigma x^\sigma,$$

where $q_j \in \mathbb{Z}$, $q_0 = 1$, and $q_\sigma = (-1)^k$. Since

$$Q(x) \sum_{n=0}^{\infty} p_A(n) x^n = 1,$$

we have

$$p_A(n + \sigma) + q_1 p_A(n + \sigma - 1) + \cdots + q_\sigma p_A(n) = 0 \tag{6.1}$$

for all $n \geq 0$. This is a homogeneous linear recurring relation of order σ.

For an integer $N \geq 2$, we can consider (6.1) as a recurring relation in the ring $\mathbb{Z}/N\mathbb{Z}$. Then there are only finitely many state vectors. Thus the sequence $\{p_A(n)\}_{n \geq 0}$ is ultimately periodic mod N, and since q_σ is a unit in $\mathbb{Z}/N\mathbb{Z}$, the sequence is periodic. The state vectors also show that the least period is at most $N^\sigma - 1$. For more precise information about the period, we can go via the generating function $F(x)$ as we did in Section 3.

Now, back to $\mathbb{Z}$. We extend the domain of p_A from the non-negative integers to all of $\mathbb{Z}$ by running (6.1) "backwards" and successively substitute $n = -1, -2, \ldots$ It follows that there is a unique extension of p_A to all of $\mathbb{Z}$ such that (6.1) holds for all $n \in \mathbb{Z}$.

Let

$$G(x) = \sum_{n=1}^{\infty} p_A(-n) x^n.$$

Then

$$G(x) = -F\left(\frac{1}{x}\right)$$

as rational functions; cf. [16, Proposition 4.2.3]. By (2.1),

$$F\left(\frac{1}{x}\right) = (-1)^k x^\sigma F(x),$$

so that

$$G(x) = (-1)^{k-1} x^\sigma F(x).$$

Hence,

$$\sum_{n=1}^{\infty} p_A(-n)x^n = (-1)^{k-1} \sum_{n=\sigma}^{\infty} p_A(n-\sigma)x^n,$$

and, as (6.2) below extends from $n \geq \sigma$ to all of $\mathbb{Z}$, we have our next theorem.

Theorem 3. *For the extended partition function $p_A(n)$ we have the reciprocity formula*

$$p_A(-n) = (-1)^{k-1} p_A(n-\sigma) \quad \textit{for all } n \in \mathbb{Z}, \tag{6.2}$$

with the supplement

$$p_A(n) = 0 \qquad \textit{if } -\sigma < n < 0. \tag{6.3}$$

Now, (2.3) holds for all $n \in \mathbb{Z}$. On the other hand, we can use (2.3) to extend the domain of p_A to all of $\mathbb{Z}$. With $n = \alpha N + r$, we have that (6.3) is equivalent to $p_A(\alpha N + r)$ having the factor $\langle N+1 \rangle_{\kappa-1}$ in $\mathbb{Q}[N]$, which is true by (3.2). Moreover, on the background of Theorem 1, the reciprocity formula (6.2) is equivalent to the polynomial $f(x)$ being reciprocal, which it is. So we could have deduced Theorem 3 directly from Theorem 1. That is, it is not difficult to see that (6.2) follows from (2.3) and the dual of (4.1). The method used in the present section is, however, quite transparent and we arrive directly at Theorem 3. Without this method it is easy to overlook results like Theorem 3.

It is well known that there is a smallest integer $g = g(A) \geq -1$, the "Frobenius number" of A, such that $p_A(n) \geq 1$ for all $n \geq g+1$. It follows by (6.2) that if $p_A(n) = 0$, then $-\sigma - g \leq n \leq g$. If $a_0 = 1$, then $p_A(n) = 0$ if and only if $-\sigma < n < 0$. In particular, for the m-ary partition function considered in the previous section, we have that the conditions $i = 1$, or $i = 2$ with (5.3), are necessary and sufficient for the congruence

$$(k-1)!\, b_{m,k}(m^{k-1}n + r) \equiv 0 \pmod{n+i}$$

to hold for some $i \in \mathbb{Z}$. In Section 5 we only proved the sufficiency.

The reciprocity formula (6.2) tells us that if we write (2.4) as

$$p_A(n) = c_{k-1}(n+\sigma/2)^{k-1} + c'_{k-2}(n+\sigma/2)^{k-2} + \cdots,$$

then, as long as the coefficients $c_{k-1}, c'_{k-2}, c'_{k-3}, \ldots$ are independent of the residue class of n mod α, every second coefficient $c'_{k-2}, c'_{k-4}, \ldots$ is equal to zero. We say more about this in the next section.

7. Asymptotics

Since $\gcd A = 1$, we have

$$p_A(n) = \frac{n^{k-1}}{(k-1)! \prod_{a\in A} a} + O(n^{k-2}) \qquad \text{as } n \to \infty. \tag{7.1}$$

This is a well-known result proven by many authors. The usual proof is based on the partial fraction decomposition of the generating function $F(x)$; cf. Netto [9], Pólya-Szegő [11, Problem 27]. However, Nathanson [7], [8, Section 15.2] proves (7.1) by induction on k.

We now give a simple arithmetic proof of (7.1). We have that $p_A(n) - p_A(n - a_s)$ is equal to the number of solutions of (1.2) with $x_i \geq 0$ and $x_s = 0$. Let $\delta = \gcd(A \setminus \{a_s\})$. If $\delta \mid n$, then $p_A(n) - p_A(n - a_s) = 0$. If $\delta \mid n$, then $p_A(n) - p_A(n - a_s) = p_B(n/\delta)$, where $B = \{a_i/\delta \mid i \neq s\}$. By (2.4) for $p_B(n/\delta)$, we have

$$p_A(n) - p_A(n - a_s) = O(n^{k-2}), \tag{7.2}$$

and this result holds whether $\delta \mid n$ or not. Since $\gcd A = 1$, there are integers u_i such that $1 = u_0a_0 + \cdots + u_{k-1}a_{k-1}$. Thus, applying (7.2) $|u_0| + \cdots + |u_{k-1}|$ times, we get

$$p_A(n) = p_A(n-1) + O(n^{k-2}). \tag{7.3}$$

If the coefficient $c_{k-1} = c_{k-1}(r)$ in (2.4) is dependent on r, there is a value of r, $0 < r < \alpha$, such that $c_{k-1}(r-1) \neq c_{k-1}(r)$. For all $n \equiv r \pmod{\alpha}$ we then have, by (2.4),

$$p_A(n) = p_A(n-1) + cn^{k-1} + O(n^{k-2}),$$

where $c = c_{k-1}(r) - c_{k-1}(r-1) \neq 0$. This contradicts (7.3). Hence the coefficient c_{k-1} in (2.4) does not depend on the residue class of n mod α.

Let us look at the coefficient of n^{k-1} in $p_A(\alpha n + r)$. Replacing n by $\alpha n + r$ in (2.4), we see that this coefficient is $c_{k-1}\alpha^{k-1}$. By (2.3), the same coefficient is $\sum_i f_{\alpha i+r}/(k-1)!$. Thus we have

$$c_{k-1}\alpha^{k-1} = \frac{1}{(k-1)!} \sum_{i=0}^{k-\kappa} f_{\alpha i+r}.$$

Summing for $r = 0, 1, \ldots, \alpha - 1$, we obtain

$$\begin{aligned} c_{k-1}\alpha^k = \sum_{r=0}^{\alpha-1} c_{k-1}\alpha^{k-1} &= \frac{1}{(k-1)!} \sum_{r=0}^{\alpha-1} \sum_{i=0}^{k-\kappa} f_{\alpha i+r} \\ &= \frac{1}{(k-1)!} \sum_{j=0}^{d} f_j = \frac{1}{(k-1)!} f(1). \end{aligned}$$

Recall that $\alpha_i = \alpha/a_i$. Using (2.2), we further get

$$c_{k-1}\alpha^k = \frac{1}{(k-1)!} \prod_{i=0}^{k-1} \sum_{j=0}^{\alpha_i - 1} 1 = \frac{1}{(k-1)!} \prod_{i=0}^{k-1} \alpha_i = \frac{\alpha^k}{(k-1)! \prod_{a\in A} a}.$$

Cancelling α^k, we obtain

$$c_{k-1} = \frac{1}{(k-1)!\prod_{a\in A} a},$$

and, by (2.4), the proof of (7.1) is complete.

There is a more refined asymptotic result for $p_A(n)$. Several authors, including [1], [3], and [5], have determined the "polynomial part" $\Phi(n)$ of $p_A(n)$, that is, the approximation of $p_A(n)$ coming from the zero $x = 1$ of $Q(x)$ (the pole $x = 1$ of $F(x)$). Almkvist [1] does this in an elegant way. He first defines symmetric polynomials $\sigma_m(x_0, \dots, x_{k-1})$ by

$$\prod_{i=0}^{k-1} \frac{x_i t/2}{\sinh(x_i t/2)} = \sum_{m=0}^{\infty} \sigma_m(x_0, \dots, x_{k-1}) t^m.$$

Thus $m!\,\sigma_m$ is a Bernoulli polynomial of higher order; see [10, Chap. 6]. Almkvist [1, Theorem 2.3] shows that

$$\Phi(n) = \frac{1}{\prod_{a\in A} a} \sum_{i=0}^{k-1} \sigma_i(a_0, \dots, a_{k-1}) \frac{(n+\sigma/2)^{k-1-i}}{(k-1-i)!}.$$

For an integer j in the interval $1 \le j \le k$, there is no zero $\xi \ne 1$ of $Q(x)$ with multiplicity v, $j \le v \le k$, if and only if $\gcd A' = 1$ for all j-subsets A' of A. In this case we have that all the coefficients $c_{k-1}, c_{k-2}, \dots, c_{j-1}$ in (2.4) are determined by the zero $x = 1$ of $Q(x)$; cf. [16, Proposition 4.4.1]. In particular, $c_{k-1}, c_{k-2}, \dots, c_{j-1}$ are independent of the residue class of n mod α. Thus we have the following result.

Theorem 4. *Let j be an integer in the interval $1 \le j \le k$. Suppose that $\gcd A' = 1$ for all j-subsets A' of A. Then we have*

$$p_A(n) = \frac{1}{\prod_{a\in A} a} \sum_{i=0}^{k-j} \sigma_i(a_0, \dots, a_{k-1}) \frac{(n+\sigma/2)^{k-1-i}}{(k-1-i)!} + O(n^{j-2})$$

as $n \to \infty$.

We have $\sigma_0 = 1$, and, in conformity with the reciprocity formula (6.2), $\sigma_m = 0$ if m is odd. Set $s_i = a_0^i + a_1^i + \cdots + a_{k-1}^i$. Then

$$\sigma_2 = -\frac{s_2}{24}, \quad \sigma_4 = \frac{5s_2^2 + 2s_4}{5760}, \quad \sigma_6 = -\frac{35s_2^3 + 42s_2 s_4 + 16 s_6}{2903040}.$$

Notice that, if the integers a_i are relatively prime *in pairs*, then Theorem 4 gives all the coefficients c_i in (2.4), with the exception of $c_0 = c_0(n)$.

In closing let us include a consequence of Theorem 4. Let $p_k(n)$ denote the number of partitions of n into at most k parts. We know that $p_k(n) = p_A(n)$ for $A = \{1, 2, \dots, k\}$. Exact expressions for $p_k(n)$ for $k \le 5$ are given in [2]. If A^* is a subset of A with $\gcd A^* > 1$, then A^* contains at most $\lfloor k/2 \rfloor$

elements. Thus we can set $j = \lfloor k/2 \rfloor + 1$ in Theorem 4 to get

$$p_k(n) = \frac{1}{k!} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \sigma_i(1, 2, \ldots, k) \frac{(n + k(k+1)/4)^{k-1-i}}{(k-1-i)!} + O(n^{\lfloor k/2 \rfloor - 1})$$

as $n \to \infty$.

The partition function $p_k(n)$ is a distinguished representative of the partition functions enumerating the partitions of n into parts in a finite set. Most results for $p_k(n)$ are valid for fixed k and variable n. An interesting but difficult problem is to find results for $p_k(n)$ valid for fixed n and variable k. Nathanson [8, p. 474] asks for an elementary proof of the unimodality of the sequence $\{p_k(n-k)\}_{1 \le k \le n}$, proven for n large by Szekeres [17, 18] using difficult analytic techniques.

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