Supplement to:
"Old Meets New: Connecting Two Infinite Families of Congruences Modulo Powers of 5 for Generalized Frobenius Partition Functions,"

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Some Definitions for Reference

We start with some preliminary definitions. We give explicit expressions for x, y, and t. We also give the expression for t in terms of x, which we will prove in the section "Level-Reduction Analysis" below, and the expression for zi=1/(1+5x), which we proved in Section 3 of the main paper:

In[29]:= $$Xq = \frac{q\,\text{QPochhammer}\left[q^2, q^2\right]\,\text{QPochhammer}\left[q^{10}, q^{10}\right]^3}{\text{QPochhammer}[q, q]^3\,\text{QPochhammer}\left[q^5, q^5\right]};$$

In[30]:=
```
Yq = (q^2 QPochhammer[q^2, q^2]^2 QPochhammer[q^4, q^4] QPochhammer[q^5, q^5] QPochhammer[q^20, q^20]^3) /
   (QPochhammer[q, q]^5 QPochhammer[q^10, q^10]^2);
```

In[31]:=
```
Tq = (q QPochhammer[q^5, q^5]^6) / (QPochhammer[q, q]^6);
```

In[11]:=
```
tR = (x + 8 x^2 + 16 x^3) / (1 + 5 x);
```

In[ ]:=
```
zi = 1 - 125 t + 120 x + 400 x^2;
```

Data and Initial values for c$\psi_{2,0}$

We give an expression for our function $A^{(0)}$ defined in our text. We have

$$U_0^{(0)}(f) = U_5\left(A^{(0)} f\right),$$
$$U_1^{(0)}(f) = U_5(f).$$

Here, for a=0,1 and b=0,1, we denote pban, as $p_a^{(b)}$ expressed in terms of t, x, y (the latter considered as indeterminates).

In[ ]:=
```
A0q = (QPochhammer[q^4, q^4]^2 QPochhammer[q^25, q^25]^2 QPochhammer[q^50, q^50]) /
   (q^4 QPochhammer[q, q]^2 QPochhammer[q^2, q^2] QPochhammer[q^100, q^100]^2);
```

In[1]:=
```
p00n = 1 + 5 t + 64 x + 600 t x + 208 x^2 + 2000 t x^2 + 4 y +
   1000 t y + 12 500 t^2 y - 992 x y - 12 000 t x y - 3520 x^2 y - 40 000 t x^2 y;
```

In[2]:=
```
p01n =
  1 + 9 t + 8 x + 64 t x + 16 x^2 + 80 t x^2 + 4 y + 8 t y + 500 t^2 y + 32 x y - 480 t x y + 64 x^2 y - 1600 t x^2 y;
```

We denote Pban, as $p_a^{(b)}$ expressed in terms of t, x, y (the latter taken as their q-Pochhammer quotients).

In[ ]:=
```
P00 = p00n /. t → Tq /. x → Xq /. y → Yq;
```

```
In[ ]:= P01 = p01n /. t → Tq /. x → Xq /. y → Yq;
```

In our relations below, we will denote pba, as $p_a^{(b)}$ as an indeterminate. We have the following command which will switch the indeterminates with their precise functions:

```
In[ ]:= sub0 =
          {p00 → P00,
           p01 → P01,
           t → Tq};
```

Now we start by writing $U_0^{(0)}(t^m)$ for m=0,-1,-2,-3,-4:

```
In[ ]:= U00 = 25 p01 - 5 t;

In[ ]:= U00tn1 = -1 + p01 / t;

In[ ]:= U00tn2 = 5^(5) t^2 + 11 × 5^(2) t + 11 - p01 (5^3 + 2 × 5 / t);

In[ ]:= U00tn3 =
          -5^(8) t^3 - 34 × 5^(5) t^2 - 51 × 5^(3) t - 119 + p01 (2 × 5^(6) t + 6 × 5^(4) + 21 × 5 / t);

In[ ]:= U00tn4 = -5^(11) t^4 + 92 × 5^(6) t^2 +
            759 × 5^(3) t + 253 × 5 - p01 (8 × 5^(7) t + 99 × 5^(4) + 44 × 5^(2) / t);
```

Next we have $U_0^{(0)}\left(p_0^{(0)}\, t^m\right)$ for m=2,-3,-4,-5,-6:

```
In[ ]:= U00p0tn2 = 5^(5) t^2 - 114 × 5^(2) t - 59 + p01 (124 × 5^(3) + 59 / t);

In[ ]:= U00p0tn3 = -5^(8) t^3 + 36 × 5^(5) t^2 + 103 × 5^(3) t + 26 + p01 (5^(6) t - 9 × 5^(4) + 7 × 5 / t);

In[ ]:= U00p0tn4 = -5^(11) t^4 - 14 × 5^(9) t^3 - 259 × 5^(6) t^2 -
            1436 × 5^(3) t - 38 × 5 + p01 (5^(9) t^2 + 122 × 5^(6) t + 211 × 5^(4) - 7 × 5 / t);

In[ ]:= U00p0tn5 = 5^(14) t^5 - 12 × 5^(11) t^4 - 9 × 5^(9) t^3 + 1494 × 5^(6) t^2 + 2366 × 5^(4) t +
            196 × 5 - p01 (5^(12) t^3 + 8 × 5^(10) t^2 + 282 × 5^(7) t + 409 × 5^(5) - 11 × 5^(2) / t);

In[ ]:= U00p0tn6 = 7 × 5^(15) t^5 + 372 × 5^(12) t^4 + 917 × 5^(10) t^3 +
            1581 × 5^(7) t^2 - 16 089 × 5^(4) t + 69 × 5^(2) - (1 / t) - p01 (96 × 5^(12) t^3 +
               13 × 5^(12) t^2 - 404 × 5^(7) t - 3152 × 5^(5) + 361 × 5^(2) / t - (1 / t^2));
```

Next we have $U_1^{(0)}(t^m)$ for m=0,-1,-2,-3,-4:

```
In[ ]:= U1 = 1;
In[ ]:= U1tn1 = -5^2 t - 6;
In[ ]:= U1tn2 = -5^(5) t^2 + 54;
In[ ]:= U1tn3 = -5^(8) t^3 - 102 × 5;
In[ ]:= U1tn4 = -5^(11) t^4 + 966 × 5;
```

Finally we have $U_1^{(0)}\left(p_1^{(0)}\, t^m\right)$ for m=-1,-2,-3,-4,-5:

```
In[ ]:= U01p1tn1 = 3 × 5^(10) t^4 + 77 × 5^(7) t^3 + 562 × 5^(4) t^2 +
           41 × 5^(3) t + 1 + p00 (5^(9) t^3 + 14 × 5^(6) t^2 + 44 × 5^(3) t + 2 × 5);
In[ ]:= U01p1tn2 = -5^(5) t^2 - 14 × 5^(2) t + 7 + 5 p00;
In[ ]:= U01p1tn3 = -5^(8) t^3 - 14 × 5^(5) t^2 - 5^4 t - 12 + 5^(4) t p00;
In[ ]:= U01p1tn4 = -5^(11) t^4 - 14 × 5^(8) t^3 - 5^(7) t^2 + 12 × 5 + 5^(7) t^2 p00;
In[ ]:= U01p1tn5 = 4 × 5^(14) t^5 + 121 × 5^(11) t^4 + 233 × 5^(9) t^3 + 738 × 5^(6) t^2 + 109 × 5^4 t -
           17 × 5^(2) - p00 (4 × 5^(10) t^3 + 14 × 5^(8) t^2 + 44 × 5^(5) t + 2 × 5^(3) - (1 / t));
```

Verification of Initial Values by Cusp Analysis

Here we give the verification of our initial relations. We emphasize that this is a routine method of checking equality between any two modular functions, and make these computations available primarily for the sake of completion.

Our first step to proving these initial relations is to understand that they are relations between modular functions---effectively, meromorphic functions on the compact Riemann surface $X_0(20)$, with some close relationships to functions on $X_0(100)$.

We first give a set of representatives for the cusps of each of these two surfaces:

In[ ]:=
```
cusps20 =
  {
   1 / 20, 1 / 10, 1 / 5, 1 / 4, 1 / 2, 1 / 1
  };
```

In[ ]:=
```
cusps100 =
  {
   1 / 100, 1 / 50, 1 / 25, 1 / 20, 1 / 10, 3 / 20,
   1 / 5, 1 / 4, 3 / 10, 7 / 20, 2 / 5, 9 / 20,
   1 / 2, 3 / 5, 7 / 10, 4 / 5, 9 / 10, 1 / 1
  };
```

We next give a formula which counts the order of a modular eta quotient at a given cusp:

In[ ]:=
```
etaCuspOrder[c_, N_, s_] :=
 Sum[
  (N / (24 c GCD[c, N / c])) s[[i]] GCD[Divisors[N][[i]], c]^2 / Divisors[N][[i]],
  {i, 1, Length[Divisors[N]]}
 ]
```

Notice that in this formula, a given eta quotient s is represented by a vector. One constructs this vector by the powers of each eta function $\eta(\delta\tau)$ in which δ is a divisor of N.

For example, one can show that our function A is modular over $X_0(100)$.

In[ ]:=
```
A0q
```

Out[ ]=

$$\frac{\text{QPochhammer}\left[q^{4}, q^{4}\right]^{2}\,\text{QPochhammer}\left[q^{25}, q^{25}\right]^{2}\,\text{QPochhammer}\left[q^{50}, q^{50}\right]}{q^{4}\,\text{QPochhammer}\left[q, q\right]^{2}\,\text{QPochhammer}\left[q^{2}, q^{2}\right]\,\text{QPochhammer}\left[q^{100}, q^{100}\right]^{2}}$$

To it we assign the vector

```
In[•]:= A0v = {-2, -1, 2, 0, 0, 0, 2, 1, -2};
```

We assign similar corresponding vectors to our functions for x, y, and t. Indeed, we assign two in these cases---one corresponding to $X_0(20)$ and one corresponding to $X_0(100)$.

```
In[•]:= Xq
```

$$\text{Out[•]=}\quad \frac{q\,\text{QPochhammer}[q^2, q^2]\,\text{QPochhammer}[q^{10}, q^{10}]^3}{\text{QPochhammer}[q, q]^3\,\text{QPochhammer}[q^5, q^5]}$$

```
In[•]:= xv = {-3, 1, 0, -1, 3, 0};

In[•]:= xv100 = {-3, 1, 0, -1, 3, 0, 0, 0, 0};

In[•]:= Yq
```

$$\text{Out[•]=}\quad \left(q^2\,\text{QPochhammer}[q^2, q^2]^2\,\text{QPochhammer}[q^4, q^4]\,\text{QPochhammer}[q^5, q^5]\,\text{QPochhammer}[q^{20}, q^{20}]^3\right) \Big/ \left(\text{QPochhammer}[q, q]^5\,\text{QPochhammer}[q^{10}, q^{10}]^2\right)$$

```
In[•]:= yv = {-5, 2, 1, 1, -2, 3};

In[•]:= yv100 = {-5, 2, 1, 1, -2, 3, 0, 0, 0};

In[•]:= Tq
```

$$\text{Out[•]=}\quad \frac{q\,\text{QPochhammer}[q^5, q^5]^6}{\text{QPochhammer}[q, q]^6}$$

```
In[•]:= tv = {-6, 0, 0, 6, 0, 0};

In[•]:= tv100 = {-6, 0, 0, 6, 0, 0, 0, 0, 0};
```

To prove our relations, we want to turn each side into a modular function with a pole only at the cusp [1/20] (equivalently, [∞]). We then compare the principal parts and constants of either side and show that they match. Because the principal parts match, the difference of the two sides will produce a function on $X_0(20)$ with no pole at all, i.e., a constant. But since the constants match also, we must indeed have the 0 function.

This finiteness condition is pivotal to the computational theory of modular functions.

Notice that we choose this cusp [∞] because it simplifies the comparison between principal parts.

We first consider the right-hand sides of our relations. Notice that our functions p are combinations of

x, x^2, t, t^2, y, and products thereof. Let us consider the order of x^2, t^2, y, at the cusps of $X_0(20)$.

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, 2 xv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=

$\{\frac{1}{20}, 2\}$
$\{\frac{1}{10}, 2\}$
$\{\frac{1}{5}, 0\}$
$\{\frac{1}{4}, 0\}$
$\{\frac{1}{2}, 0\}$
$\{1, -4\}$

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, yv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=

$\{\frac{1}{20}, 2\}$
$\{\frac{1}{10}, 0\}$
$\{\frac{1}{5}, 0\}$
$\{\frac{1}{4}, 1\}$
$\{\frac{1}{2}, 0\}$
$\{1, -3\}$

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, 2 tv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=

$\left\{\frac{1}{20}, 2\right\}$

$\left\{\frac{1}{10}, 2\right\}$

$\left\{\frac{1}{5}, 8\right\}$

$\left\{\frac{1}{4}, -2\right\}$

$\left\{\frac{1}{2}, -2\right\}$

$\{1, -8\}$

We also have our generators multiplied by t^m with m going from -2 to 3.

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, -2 tv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=

$\left\{\frac{1}{20}, -2\right\}$

$\left\{\frac{1}{10}, -2\right\}$

$\left\{\frac{1}{5}, -8\right\}$

$\left\{\frac{1}{4}, 2\right\}$

$\left\{\frac{1}{2}, 2\right\}$

$\{1, 8\}$

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, 3 tv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=
$\left\{\frac{1}{20}, 3\right\}$
$\left\{\frac{1}{10}, 3\right\}$
$\left\{\frac{1}{5}, 12\right\}$
$\left\{\frac{1}{4}, -3\right\}$
$\left\{\frac{1}{2}, -3\right\}$
$\{1, -12\}$

We need to multiply our right-hand sides by a function which will kill all of these poles except that at [1/20]. One suitable function we will denote as μ_0 and define as

In[ ]:=
$$\text{mu0} = \frac{\text{QPochhammer}\left[q^4, q^4\right]^4 \text{QPochhammer}\left[q^{10}, q^{10}\right]^8}{q^6 \text{QPochhammer}\left[q^{20}, q^{20}\right]^{12}};$$

In[ ]:=
```
mu0v = {0, 0, 4, 0, 8, -12};
```

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, mu0v]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=
$\left\{\frac{1}{20}, -6\right\}$
$\left\{\frac{1}{10}, 1\right\}$
$\left\{\frac{1}{5}, 1\right\}$
$\left\{\frac{1}{4}, 2\right\}$
$\left\{\frac{1}{2}, 1\right\}$
$\{1, 1\}$

Notice that by multiplying the right hand side of each relation by ${\mu_0}^{27}$, we can kill every pole except that at [1/20] (Actually, this is not optimal. We are being crude for the sake of simplicity of exposition, but a clever reader can easily optimize these calculations).

Next, we need to examine the left-hand sides of our functions. These are of the form $U_5\left(A^i\, y^j\, x^k\, t^m\right)$, with i=0,1, j=0,1, k=0,1,2, and m ranging from -6 to 2. The interior functions are modular over $X_0(100)$. Applying U_5 gives us a function over $X_0(20)$.

Let us look at the possible orders of $U_5\left(A^i\, y^j\, x^k\, t^m\right)$. If we look at the possible combinations of A, y, x, t, by applying A to our module generators, we get

In[ ]:=
```
A p00n // Expand
```

Out[ ]=
$A + 5\,A\,t + 64\,A\,x + 600\,A\,t\,x + 208\,A\,x^2 + 2000\,A\,t\,x^2 + 4\,A\,y + 1000\,A\,t\,y + 12\,500\,A\,t^2\,y - 992\,A\,x\,y - 12\,000\,A\,t\,x\,y - 3520\,A\,x^2\,y - 40\,000\,A\,t\,x^2\,y$

In[ ]:=
```
A p01n // Expand
```

Out[ ]=
$A + 9\,A\,t + 8\,A\,x + 64\,A\,t\,x + 16\,A\,x^2 + 80\,A\,t\,x^2 + 4\,A\,y + 8\,A\,t\,y + 500\,A\,t^2\,y + 32\,A\,x\,y - 480\,A\,t\,x\,y + 64\,A\,x^2\,y - 1600\,A\,t\,x^2\,y$

We can work out estimations for the order of each monomial after application of U_5 using Theorem 47 from Radu. If we define

```
In[ ]:= p[y_, M_, r_, m_] :=
  Min[
   Table[
    1/24 Sum[
      r[[i]] (
        GCD[
           Divisors[M][[i]] (y[[1]][[1]] + GCD[m^2 - 1, 24] l y[[2]][[1]]), m y[[2]][[1]]
          ]^2 / (Divisors[M][[i]] m)
       ),
      {i, 1, Length[Divisors[M]]}
     ],
    {l, 0, m - 1}
   ]
  ]
```

then the order of an eta quotient over $X_0(100)$ at a given cusp of $X_0(20)$ after application of U_5 is

```
20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v, 5]
```

We therefore examine the possible orders that our monomials will create.

```
In[ ]:= Column[
  Table[
   {
    cusps20[[j]],
```

```
N[20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v + tv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v + xv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + tv100 + xv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v + 2 xv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + tv100 + 2 xv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20] p[repMat[20][[j]], 100, A0v + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + tv100 + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + 2 tv100 + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + xv100 + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + tv100 + xv100 + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + 2 xv100 + yv100, 5]],
N[20/GCD[Denominator[cusps20[[j]]]^2, 20]
  p[repMat[20][[j]], 100, A0v + tv100 + 2 xv100 + yv100, 5]],
etaCuspOrder[Denominator[cusps20[[j]]], 20, 63 mu0v]
},
{j, 1, Length[repMat[20]]}
]
]
```

Out[ ]=
$\left\{\frac{1}{20}, -0.8, -0.6, -0.6, -0.4, -0.4, -0.2, -0.4, -0.2, 0., -0.2, 0., 0., 0.2, -378\right\}$
$\left\{\frac{1}{10}, 0.4, 0.6, 0.6, 0.8, 0.8, 1., 0.4, 0.6, 0.8, 0.6, 0.8, 0.8, 1., 63\right\}$
$\left\{\frac{1}{5}, 1.6, 2.4, 1.6, 2.4, 1.6, 2.4, 1.6, 2.4, 3.2, 1.6, 2.4, 1.6, 2.4, 63\right\}$
$\left\{\frac{1}{4}, 0., -1., 1., -1., 2., -1., 2., 3., -1., 3., 4., 4., 4., 126\right\}$
$\left\{\frac{1}{2}, -2., -7., -2., -7., -2., -7., -2., -7., -12., -2., -7., -2., -7., 63\right\}$
$\{1, -8., -28., -18., -38., -28., -48., -23., -43., -63., -33., -53., -43., -63., 63\}$

We see that we will have to multiply by μ_0^{63} to guarantee that all poles away from [1/20] have been killed (again, being relatively crude here).

So we can multiply both sides of all of our relations by μ_0^{63} and be guaranteed to have a comparison of functions with a pole only at [1/20]. Matching the principal parts and constants will then constitute a full equality.

We can move μ_0 inside of the U_5 operator; doing so dilates the powers of q by a factor of 5:

$$\textbf{mu5} = \frac{\text{QPochhammer}\left[q^{20}, q^{20}\right]^4 \text{QPochhammer}\left[q^{50}, q^{50}\right]^8}{q^{30}\,\text{QPochhammer}\left[q^{100}, q^{100}\right]^{12}};$$

```
mu5v = {0, 0, 0, 0, 0, 4, 0, 8, -12};
```

With this in mind, we create a table of our left-hand sides before application of U_5 (again, with μ_0 dilated). Notice that we neglect five relations in which U_5 is simply applied to negative powers of t, since these were already proved in Paule--Radu. That leaves us with 15 relations to check.

In[ ]:=
```
Lv =
  {
   mu5^(63) A0q,
   mu5^(63) A0q / Tq,
   mu5^(63) A0q / Tq^2,
   mu5^(63) A0q / Tq^3,
   mu5^(63) A0q / Tq^4,
   mu5^(63) A0q P00 / Tq^2,
   mu5^(63) A0q P00 / Tq^3,
   mu5^(63) A0q P00 / Tq^4,
   mu5^(63) A0q P00 / Tq^5,
   mu5^(63) A0q P00 / Tq^6,
   mu5^(63) P01 / Tq,
   mu5^(63) P01 / Tq^2,
   mu5^(63) P01 / Tq^3,
   mu5^(63) P01 / Tq^4,
   mu5^(63) P01 / Tq^5
  };
```

We next create our right-hand side:

In[ ]:=
```
Rv =
  {
   mu0^(63) U00 /. sub0,
   mu0^(63) U00tn1 /. sub0,
   mu0^(63) U00tn2 /. sub0,
   mu0^(63) U00tn3 /. sub0,
   mu0^(63) U00tn4 /. sub0,
   mu0^(63) U00p0tn2 /. sub0,
   mu0^(63) U00p0tn3 /. sub0,
   mu0^(63) U00p0tn4 /. sub0,
   mu0^(63) U00p0tn5 /. sub0,
   mu0^(63) U00p0tn6 /. sub0,
   mu0^(63) U01p1tn1 /. sub0,
   mu0^(63) U01p1tn2 /. sub0,
   mu0^(63) U01p1tn3 /. sub0,
   mu0^(63) U01p1tn4 /. sub0,
   mu0^(63) U01p1tn5 /. sub0
  };
```

We need to be careful in defining the order of the principal part of a given function:

```
In[ ]:= pord[f_] :=
 If[
  -Exponent[Series[f, {q, 0, 1}], q, Min] == -Infinity,
  0,
  -Exponent[Series[f, {q, 0, 1}], q, Min]
 ]
```

Now we build a function which will take the principal parts and constants of both sides and subtract them. We are only interested in present in properly computing the coefficients up to our constant term (one can of course extend the acceptable range of these computations by increasing the number of terms computed in the series expansions).

```
In[ ]:= TestM[A_, B_] :=
 Module[
  {
   aS = Series[A, {q, 0, 0}],
   uaS = 1,
   ubS = Series[
      B,
      {q, 0, 380}
     ]
  },
  uaS =
   Series[
    Sum[
     Coefficient[aS, q, 5 n] q^n,
     {n, -pord[aS], 380}
    ], {q, 0, 0}
   ]
  ;
  uaS - ubS // Expand
 ]
```

We finally check our 15 relations. Here we check them one at a time, and give the timing that was necessary for our computer (a Lenovo Thinkpad T14s) to finish them.

```
In[ ]:= Timing[TestM[Lv[[1]], Rv[[1]]]]
Out[ ]= {32.6563, O[q]^1}

In[ ]:= Timing[TestM[Lv[[2]], Rv[[2]]]]
Out[ ]= {69., O[q]^1}

In[ ]:= Timing[TestM[Lv[[3]], Rv[[3]]]]
Out[ ]= {75.8281, O[q]^1}

In[ ]:= Timing[TestM[Lv[[4]], Rv[[4]]]]
Out[ ]= {79.7344, O[q]^1}

In[ ]:= Timing[TestM[Lv[[5]], Rv[[5]]]]
Out[ ]= {82.0469, O[q]^1}

In[ ]:= Timing[TestM[Lv[[6]], Rv[[6]]]]
Out[ ]= {324.891, O[q]^1}

In[ ]:= Timing[TestM[Lv[[7]], Rv[[7]]]]
Out[ ]= {336.063, O[q]^1}

In[ ]:= Timing[TestM[Lv[[8]], Rv[[8]]]]
Out[ ]= {349.609, O[q]^1}

In[ ]:= Timing[TestM[Lv[[9]], Rv[[9]]]]
Out[ ]= {312.563, O[q]^1}

In[ ]:= Timing[TestM[Lv[[10]], Rv[[10]]]]
Out[ ]= {370.172, O[q]^1}

In[ ]:= Timing[TestM[Lv[[11]], Rv[[11]]]]
Out[ ]= {241.625, O[q]^1}

In[ ]:= Timing[TestM[Lv[[12]], Rv[[12]]]]
Out[ ]= {252.609, O[q]^1}

In[ ]:= Timing[TestM[Lv[[13]], Rv[[13]]]]
Out[ ]= {271.625, O[q]^1}

In[ ]:= Timing[TestM[Lv[[14]], Rv[[14]]]]
Out[ ]= {281.656, O[q]^1}
```

In[ ]:= `Timing[TestM[Lv[[15]], Rv[[15]]]]`

Out[ ]= $\{325.953, O[q]^{1}\}$

Level-Reduction Analysis

The initial relations given above match exactly those in Groups I-IV of the Appendix of Paule--Radu. The only difference is in the form of the module generators.

These generators may be represented as

In[3]:= $p11n = t + 25 t^2 - 8 t x - 4 y - 8 t y - 500 t^2 y - 32 x y + 480 t x y - 64 x^2 y + 1600 t x^2 y;$

In[4]:= $p10n = 85 t + 625 t^2 - 24 x - 64 x^2 - 4 y - 1000 t y - 12500 t^2 y + 992 x y + 12000 t x y + 3520 x^2 y + 40000 t x^2 y;$

Here, by pba, we mean $p_a^{(b)}$ as written in our main paper.

If we take $p_a^{(b)} + p_a^{(1-b)}$, we get

In[ ]:= `p01n + p11n // Expand`

Out[ ]= $1 + 10 t + 25 t^2 + 8 x + 56 t x + 16 x^2 + 80 t x^2$

In[ ]:= `p00n + p10n // Expand`

Out[ ]= $1 + 90 t + 625 t^2 + 40 x + 600 t x + 144 x^2 + 2000 t x^2$

Notice that the contribution of y vanishes altogether. Next we examine x and notice that it is modular over $X_0(10)$ (indeed, it is a Hauptmodul at [0], as can be shown by examining its orders at all cusps).

In[ ]:= **Xq**

Out[ ]= $$\frac{q\,\text{QPochhammer}[q^2, q^2]\,\text{QPochhammer}[q^{10}, q^{10}]^3}{\text{QPochhammer}[q, q]^3\,\text{QPochhammer}[q^5, q^5]}$$

Also, we have t as a rational combination of powers of x. To quickly prove this, we note that

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, {-5, 5, 0, 1, -1, 0}],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, xv]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=
$\left\{\frac{1}{20}, 0, 1\right\}$
$\left\{\frac{1}{10}, 0, 1\right\}$
$\left\{\frac{1}{5}, 0, 0\right\}$
$\left\{\frac{1}{4}, 1, 0\right\}$
$\left\{\frac{1}{2}, 1, 0\right\}$
$\{1, -2, -2\}$

Therefore, the function which we denote by z, or Zq in code,

In[ ]:= **Zq** = $$\frac{\text{QPochhammer}[q^2, q^2]^5\,\text{QPochhammer}[q^5, q^5]}{\text{QPochhammer}[q, q]^5\,\text{QPochhammer}[q^{10}, q^{10}]};$$

must be a scalar multiple of x plus some complex number. We quickly find that

In[ ]:=
```
Series[
 Zq - (1 + 5 Xq),
 {q, 0, 100}
]
```

Out[ ]= $O[q]^{101}$

Now we compare the orders of z with those of t:

In[ ]:=
```
Column[
 Table[
  {
   cusps20[[k]],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, {-5, 5, 0, 1, -1, 0}],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, tv],
   etaCuspOrder[Denominator[cusps20[[k]]], 20, 6 mu0v]
  },
  {k, 1, Length[cusps20]}
 ]
]
```

Out[ ]=

$\left\{\frac{1}{20}, 0, 1, -36\right\}$

$\left\{\frac{1}{10}, 0, 1, 6\right\}$

$\left\{\frac{1}{5}, 0, 4, 6\right\}$

$\left\{\frac{1}{4}, 1, -1, 12\right\}$

$\left\{\frac{1}{2}, 1, -1, 6\right\}$

$\{1, -2, -4, 6\}$

So zt must be some combination of powers of x and y by powers of x. One guesses from

In[ ]:=
```
Series[
 Zq Tq - (Xq + 8 Xq^2 + 16 Xq^3),
 {q, 0, 100}
]
```

Out[ ]= $O[q]^{101}$

that t must have the form we defined above:

In[ ]:=
```
tR
```

Out[ ]=
$$\frac{x + 8 x^2 + 16 x^3}{1 + 5 x}$$

We prove this by multiplying by a sufficient power of μ_0 to push everything up to the cusp at [1/20], and checking to see that the principal parts and constants are indeed killed:

In[ ]:=
```
Series[
 (mu0^(6)) Zq Tq - (mu0^(6)) (Xq + 8 Xq^2 + 16 Xq^3),
 {q, 0, 100}
]
```

Out[ ]=
$$O[q]^{101}$$

Therefore, we have shown that t is a rational polynomial in x alone, and x is modular over $X_0(10)$. So indeed, $p_a^{(b)} + p_a^{(1-b)}$ is modular over the same; we have reduced the level of the function.

Computations for the Atkin--Lehner Involution

Here we will walk through the computations needed to prove Theorem 5.2. We will only consider the computations for $c\psi_{2,0}(n)$, since Paule and Radu had already considered $c\psi_{2,1}(n)$. In any case, the latter relations may be demonstrated in an analogous manner.

First, we note that, as discussed in Section 3, our relevant functions may be built as elements of a rank 6 $\mathbb{Z}$[t] module with generators 1, x, x^2, y, $x\,y$, $x^2\,y$. As such, we build a function which will take any polynomial in x and reduce it so that all powers of x are 2 or less.

```
In[ ]:= tReduce[f_] :=
 Module[
  {
   fn = f,
   j = Exponent[f, x]
  },
  While[
   j ≥ 3,
   Set[
    fn, Sum[x^k Coefficient[fn, x, k], {k, 0, 2}] + Sum[x^(k - 3) (1/16 (t + 5 x t - x - 8 x^2))
        Coefficient[fn, x, k], {k, 3, Exponent[fn, x]}] // Expand
   ];
   Set[j, Exponent[fn, x]]
  ];
  fn
 ]
```

Similarly, we will reduce any power of y greater than 2 to a polynomial in x and powers of t in which the power of y is either 0 or 1.

```
In[ ]:= yReduce[f_] :=
  Module[
   {
    fn = f,
    j = Exponent[f, y]
   },
   While[
    j ≥ 2,
    Set[
     fn,
     Sum[y^k Coefficient[fn, y, k], {k, 0, 1}] + Sum[y^(k - 2) (1/4 (x^2 + 5 x^3 - y - 6 x y))
         Coefficient[fn, y, k], {k, 2, Exponent[fn, y]}] // Expand
    ];
    Set[j, Exponent[fn, y]]
   ];
   fn
  ]
```

We showed in Theorem 5.1 that x, y, z=1+5x, all have certain transformations via our chosen Atkin--Lehner involution. We can build up the formula of Theorem 5.1 using the effects of the involution on x, 1/(1+5x), and y.

Notice that by ziw, we mean the effect of the involution on 1/(1+5x).

```
In[ ]:= xw = (1 + 10 x + 20 x^2 + 4 y)/(1 + 5 x)^2;

In[ ]:= ziw = (1 + 5 x + 5 y)/(1 + 5 x);

In[ ]:= yw = (1 + 14 x + 40 x^2 + (20 + 80 x) y)/(1 + 5 x)^3;

In[ ]:= AL[m_, n_, j_] :=
  4^(n - 2 j - m) (-1)^(m + n) xw^m ziw^n yw^j
```

We want to convert this into how the involution affects polynomials in x, y, t.

For example, if we take xw and recall that we have zi=1/(1+5x), then we have

In[ ]:= $(1+10x+20x^2+4y)$ zi^2 // Expand

Out[ ]= $1-250t+15625t^2+250x-32500tx+156250t^2x+17620x^2-405000tx^2+312500t^2x^2+252800x^3-1600000tx^3+1424000x^4-2000000tx^4+3520000x^5+3200000x^6+4y-1000ty+62500t^2y+960xy-120000txy+60800x^2y-400000tx^2y+384000x^3y+640000x^4y$

This expression contains powers of x which are greater than 2. So we apply tReduce. This is the effect of the involution on x (not yet counting associated powers of -1 and 2).

In[ ]:= tReduce[$(1+10x+20x^2+4y)$ zi^2 // Expand]

Out[ ]= $1-200t-3125t^2+200x+3000tx+720x^2+10000tx^2+4y+3000ty+62500t^2y-3040xy-60000txy-11200x^2y-200000tx^2y$

In[ ]:= xW = $1-200t-3125t^2+200x+3000tx+720x^2+10000tx^2+4y+3000ty+62500t^2y-3040xy-60000txy-11200x^2y-200000tx^2y$;

We can get the effect of the involution on 1/(1+5x) in a similar manner:

In[ ]:= tReduce[(1 + 5 x + 5 y) zi // Expand]

Out[ ]= $1+5y-625ty+600xy+2000x^2y$

In[ ]:= ziW = $1+5y-625ty+600xy+2000x^2y$;

Finally, we do the same with y:

In[ ]:= tReduce[$(1+14x+40x^2+(20+80x)y)$ zi^3 // Expand]

Out[ ]= $1+375t+34375t^2+390625t^3-376x-34000tx-375000t^2x-1520x^2-120000tx^2-1250000t^2x^2+20y-5500ty-562500t^2y-7812500t^3y+5280xy+560000txy+7500000t^2xy+20800x^2y+2000000tx^2y+25000000t^2x^2y$

In[ ]:= yW = 1 + 375 t + 34 375 t^2 + 390 625 t^3 - 376 x - 34 000 t x - 375 000 t^2 x - 1520 x^2 - 120 000 t x^2 - 1 250 000 t^2 x^2 + 20 y - 5500 t y - 562 500 t^2 y - 7 812 500 t^3 y + 5280 x y + 560 000 t x y + 7 500 000 t^2 x y + 20 800 x^2 y + 2 000 000 t x^2 y + 25 000 000 t^2 x^2 y;

There are various ways of getting the effect on t (including simple direct computation). One way is to examine how we can express t in terms of x:

In[ ]:= tR

Out[ ]= $\frac{x + 8 x^2 + 16 x^3}{1 + 5 x}$

We now apply our formula AL[m,n,j] to each rational monomial in this expression:

In[ ]:= AL[1, 1, 0] + 8 AL[2, 1, 0] + 16 AL[3, 1, 0] // Together

Out[ ]= $\frac{(1 + 10 x + 20 x^2 + 4 y)\ (1 + 5 x + 5 y)\ (25 x^4 - 40 x^2 y + 16 y^2)}{(1 + 5 x)^7}$

Multiplying this out will induce large powers of x and y, together with a denominator in x. So instead, we replace 1/(1+5x) with zi, then we apply yReduce and tReduce. This gives us the effect of the involution on t:

In[ ]:= tReduce[yReduce[(1 + 10 x + 20 x^2 + 4 y) (1 + 5 x + 5 y) (25 x^4 - 40 x^2 y + 16 y^2) zi^7]]

Out[ ]= $136 t + 14000 t^2 + 312500 t^3 + 1953125 t^4 - 136 x - 13880 t x - 305000 t^2 x - 1875000 t^3 x - 528 x^2 - 50000 t x^2 - 1050000 t^2 x^2 - 6250000 t^3 x^2 - 1980 t y - 225000 t^2 y - 5625000 t^3 y - 39062500 t^4 y + 1984 x y + 223200 t x y + 5500000 t^2 x y + 37500000 t^3 x y + 7680 x^2 y + 808000 t x^2 y + 19000000 t^2 x^2 y + 125000000 t^3 x^2 y$

In[ ]:= tW = 136 t + 14 000 t^2 + 312 500 t^3 + 1 953 125 t^4 - 136 x - 13 880 t x - 305 000 t^2 x - 1 875 000 t^3 x - 528 x^2 - 50 000 t x^2 - 1 050 000 t^2 x^2 - 6 250 000 t^3 x^2 - 1980 t y - 225 000 t^2 y - 5 625 000 t^3 y - 39 062 500 t^4 y + 1984 x y + 223 200 t x y + 5 500 000 t^2 x y + 37 500 000 t^3 x y + 7680 x^2 y + 808 000 t x^2 y + 19 000 000 t^2 x^2 y + 125 000 000 t^3 x^2 y;

So now we have an alternative expression for how the involution affects a polynomial in x, t, y:

In[ ]:=
```
ALt[m_, n_, j_] :=
 4^(-2 j - m) (-1)^(m) tReduce[yReduce[xW^m tW^n yW^j]]
```

$c\psi_{2,0}(n)$: Odd Index, Progression 4n+2

We will use ALt to study the effect of the involution (henceforth called W) on the arithmetic progressions of $c\psi_{2,0}(n)$. As shown in Section 5, we only need consider $p_1^{(0)}$, and the necessary behavior for $p_0^{(0)}$ will follow.

We begin by reminding ourselves of $p_1^{(0)}$:

In[ ]:=
```
p01n
```

Out[ ]= $1 + 9 t + 8 x + 64 t x + 16 x^2 + 80 t x^2 + 4 y + 8 t y + 500 t^2 y + 32 x y - 480 t x y + 64 x^2 y - 1600 t x^2 y$

We apply W to $p_1^{(0)}$ and note its form:

In[ ]:=
```
p01w =
 ALt[0, 0, 0] + 9 ALt[0, 1, 0] + 8 ALt[1, 0, 0] + 64 ALt[1, 1, 0] +
   16 ALt[2, 0, 0] + 80 ALt[2, 1, 0] + 4 ALt[0, 0, 1] + 8 ALt[0, 1, 1] + 500 ALt[0, 2, 1] +
   32 ALt[1, 0, 1] - 480 ALt[1, 1, 1] + 64 ALt[2, 0, 1] - 1600 ALt[2, 1, 1] // Together
```

Out[ ]= $5200 t + 2\,674\,400 t^2 + 320\,787\,500 t^3 + 15\,859\,765\,625 t^4 + 393\,457\,031\,250 t^5 + 5\,200\,195\,312\,500 t^6 + 35\,095\,214\,843\,750 t^7 + 95\,367\,431\,640\,625 t^8 - 5200 x - 2\,669\,488 t x - 318\,698\,000 t^2 x - 15\,660\,250\,000 t^3 x - 385\,890\,625\,000 t^4 x - 5\,064\,453\,125\,000 t^5 x - 33\,935\,546\,875\,000 t^6 x - 91\,552\,734\,375\,000 t^7 x - 20\,512 x^2 - 10\,132\,640 t x^2 - 1\,170\,740\,000 t^2 x^2 - 56\,027\,500\,000 t^3 x^2 - 1\,350\,781\,250\,000 t^4 x^2 - 17\,402\,343\,750\,000 t^5 x^2 - 114\,746\,093\,750\,000 t^6 x^2 - 305\,175\,781\,250\,000 t^7 x^2 - 75\,964 t y - 39\,945\,500 t^2 y - 5\,031\,000\,000 t^3 y - 262\,117\,187\,500 t^4 y - 6\,843\,750\,000\,000 t^5 y - 94\,970\,703\,125\,000 t^6 y - 671\,386\,718\,750\,000 t^7 y - 1\,907\,348\,632\,812\,500 t^8 y + 75\,968 x y + 39\,873\,760 t x y + 4\,999\,640\,000 t^2 x y + 258\,945\,000\,000 t^3 x y + 6\,715\,937\,500\,000 t^4 x y + 92\,539\,062\,500\,000 t^5 x y + 649\,414\,062\,500\,000 t^6 x y + 1\,831\,054\,687\,500\,000 t^7 x y + 299\,520 x^2 y + 151\,489\,600 t x^2 y + 18\,398\,000\,000 t^2 x^2 y + 928\,150\,000\,000 t^3 x^2 y + 23\,546\,875\,000\,000 t^4 x^2 y + 318\,359\,375\,000\,000 t^5 x^2 y + 2\,197\,265\,625\,000\,000 t^6 x^2 y + 6\,103\,515\,625\,000\,000 t^7 x^2 y$

We construct the list of monomials in this expression:

In[ ]:=
```
mLp01w =
 MonomialList[p01w]
```

Out[ ]= {-1 907 348 632 812 500 t^8 y, 95 367 431 640 625 t^8, 6 103 515 625 000 000 t^7 x^2 y, -305 175 781 250 000 t^7 x^2, 1 831 054 687 500 000 t^7 x y, -91 552 734 375 000 t^7 x, -671 386 718 750 000 t^7 y, 35 095 214 843 750 t^7, 2 197 265 625 000 000 t^6 x^2 y, -114 746 093 750 000 t^6 x^2, 649 414 062 500 000 t^6 x y, -33 935 546 875 000 t^6 x, -94 970 703 125 000 t^6 y, 5 200 195 312 500 t^6, 318 359 375 000 000 t^5 x^2 y, -17 402 343 750 000 t^5 x^2, 92 539 062 500 000 t^5 x y, -5 064 453 125 000 t^5 x, -6 843 750 000 000 t^5 y, 393 457 031 250 t^5, 23 546 875 000 000 t^4 x^2 y, -1 350 781 250 000 t^4 x^2, 6 715 937 500 000 t^4 x y, -385 890 625 000 t^4 x, -262 117 187 500 t^4 y, 15 859 765 625 t^4, 928 150 000 000 t^3 x^2 y, -56 027 500 000 t^3 x^2, 258 945 000 000 t^3 x y, -15 660 250 000 t^3 x, -5 031 000 000 t^3 y, 320 787 500 t^3, 18 398 000 000 t^2 x^2 y, -1 170 740 000 t^2 x^2, 4 999 640 000 t^2 x y, -318 698 000 t^2 x, -39 945 500 t^2 y, 2 674 400 t^2, 151 489 600 t x^2 y, -10 132 640 t x^2, 39 873 760 t x y, -2 669 488 t x, -75 964 t y, 5200 t, 299 520 x^2 y, -20 512 x^2, 75 968 x y, -5200 x}

We also construct the list of coefficients:

In[ ]:=
```
mLp01wC = {-1 907 348 632 812 500, 95 367 431 640 625, 6 103 515 625 000 000, -305 175 781 250 000,
   1 831 054 687 500 000, -91 552 734 375 000, -671 386 718 750 000, 35 095 214 843 750,
   2 197 265 625 000 000, -114 746 093 750 000, 649 414 062 500 000, -33 935 546 875 000,
   -94 970 703 125 000, 5 200 195 312 500, 318 359 375 000 000, -17 402 343 750 000,
   92 539 062 500 000, -5 064 453 125 000, -6 843 750 000 000, 393 457 031 250, 23 546 875 000 000,
   -1 350 781 250 000, 6 715 937 500 000, -385 890 625 000, -262 117 187 500, 15 859 765 625,
   928 150 000 000, -56 027 500 000, 258 945 000 000, -15 660 250 000, -5 031 000 000, 320 787 500,
   18 398 000 000, -1 170 740 000, 4 999 640 000, -318 698 000, -39 945 500, 2 674 400, 151 489 600,
   -10 132 640, 39 873 760, -2 669 488, -75 964, 5200, 299 520, -20 512, 75 968, -5200};
```

Of course, we have already examined t, y, x by considering their associated vectors, which we remind the reader of here:

In[ ]:=
```
tv
```

Out[ ]= {-6, 0, 0, 6, 0, 0}

In[ ]:=
```
yv
```
Out[ ]=
```
{-5, 2, 1, 1, -2, 3}
```
In[ ]:=
```
xv
```
Out[ ]=
```
{-3, 1, 0, -1, 3, 0}
```

Now we are interested in the arithmetic progression 4n+2. The functions t, x, each begin with q^1, and y begins with q^2. So the associated progression 4n+2 in a given monomial with powers of t, x, y can be given as

In[ ]:=
```
rem2[nt_, nx_, ny_] :=
 Mod[2 - (nt + nx + 2 ny), 4]
```

With nt, nx, ny denoting the powers of t, x, y (respectively) in the monomial. Thus, rem2 will give us the key progression to examine mod 4 in the given associated infinite products for t, x, y.

Now we must appeal to software that has already been developed to study arithmetic progressions in eta quotients. The user will need the software RaduRK to explicitly work through the following steps (though we will illustrate them as we go).

```
<< RaduRK`
```

math4ti2: Mathematica interface to 4ti2 (http://www.4ti2.de/)

RaduRK: Ramanujan--Kolberg Program Version 2.8

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Johannes Kepler Universität, Linz

The first thing we will do is determine the associated modular curve that we will have to work on in examining the progression 4n+2. We build a modified version of minN for this purpose. Notice that we do not just have the power of t, x, and y in the given monomial, but also the presence of $\phi(-q^5)$ (here indicated by the vector (0,0,0,2,-1,0) indexed over the powers of 20).

In[ ]:=
```
MN2[f_] :=
 minN[
  20,
  Exponent[f, t] tv + Exponent[f, x] xv + Exponent[f, y] yv + {0, 0, 0, 2, -1, 0},
  4,
  rem2[Exponent[f, t], Exponent[f, x], Exponent[f, y]]
 ]
```

Now if we examine the necessary level of the modular curve for each monomial, we have a fortunate pattern:

In[ ]:=
```
Column[
 Table[
  {j, MN2[mLp01w[[j]]]},
  {j, 1, Length[mLp01w]}
 ]
]
```

Out[•]=

```
{1, 40}
{2, 40}
{3, 40}
{4, 40}
{5, 40}
{6, 40}
{7, 40}
{8, 40}
{9, 40}
{10, 40}
{11, 40}
{12, 40}
{13, 40}
{14, 40}
{15, 40}
{16, 40}
{17, 40}
{18, 40}
{19, 40}
{20, 40}
{21, 40}
{22, 40}
{23, 40}
{24, 40}
{25, 40}
{26, 40}
{27, 40}
{28, 40}
{29, 40}
{30, 40}
{31, 40}
{32, 40}
{33, 40}
{34, 40}
{35, 40}
{36, 40}
{37, 40}
{38, 40}
{39, 40}
{40, 40}
{41, 40}
{42, 40}
{43, 40}
{44, 40}
{45, 40}
{46, 40}
{47, 40}
{48, 40}
```

We will therefore work over the modular curve $X_0(40)$. This has genus 3 (this can be computed through formulae in Diamond--Shurman, or through consulting OEIS sequence A001617). As such, we can build an algebra basis for the functions over this curve with a pole only at [∞] with functions which have orders -4, -5, -6, -7 at [∞]. Again appealing to RaduRK, we can use the algebra commands AB and etaGenerators to build the following algebra basis (which may be checked directly through the etaCuspOrder function we defined in the previous section):

```
In[ ]:= ab40 =
  {
   QPochhammer[q^4, q^4]^3 QPochhammer[q^20, q^20] / (q^4 QPochhammer[q^8, q^8] QPochhammer[q^40, q^40]^3),
   {
    1,
    (QPochhammer[q^2, q^2]^3 QPochhammer[q^5, q^5] QPochhammer[q^20, q^20]^2) /
     (q^5 QPochhammer[q, q] QPochhammer[q^10, q^10] QPochhammer[q^40, q^40]^4),
    QPochhammer[q^8, q^8] QPochhammer[q^10, q^10]^5 / (q^6 QPochhammer[q^2, q^2] QPochhammer[q^40, q^40]^5),
    (QPochhammer[q^4, q^4]^4 QPochhammer[q^20, q^20]^8) /
     (q^7 QPochhammer[q^2, q^2]^2 QPochhammer[q^10, q^10]^2 QPochhammer[q^40, q^40]^8)
   }
  };
```

The associated Ramanujan--Kolberg identity for taking the associated progression may consist of multiple different progressions, which would substantially complicate our approach. We can show that this will not be a problem by building a modified Pmr function:

In[ ]:=
```
Pmr2[f_] :=
 Pmr[
  40,
  20,
  Exponent[f, t] tv + Exponent[f, x] xv + Exponent[f, y] yv + {0, 0, 0, 2, -1, 0},
  4,
  rem2[Exponent[f, t], Exponent[f, x], Exponent[f, y]]
 ]
```

If we examine every monomial we have with its associated progression, we find that we will never have multiple different progressions, since the length of Pmr2 is always just 1:

In[ ]:=
```
Column[
 Table[
  {j, Length[Pmr2[mLp01w[[j]]]]},
  {j, 1, Length[mLp01w]}
 ]
]
```

Out[ ]=

```
{1, 1}
{2, 1}
{3, 1}
{4, 1}
{5, 1}
{6, 1}
{7, 1}
{8, 1}
{9, 1}
{10, 1}
{11, 1}
{12, 1}
{13, 1}
{14, 1}
{15, 1}
{16, 1}
{17, 1}
{18, 1}
{19, 1}
{20, 1}
{21, 1}
{22, 1}
{23, 1}
{24, 1}
{25, 1}
{26, 1}
{27, 1}
{28, 1}
{29, 1}
{30, 1}
{31, 1}
{32, 1}
{33, 1}
{34, 1}
{35, 1}
{36, 1}
{37, 1}
{38, 1}
{39, 1}
{40, 1}
{41, 1}
{42, 1}
{43, 1}
{44, 1}
{45, 1}
{46, 1}
{47, 1}
{48, 1}
```

We will have to multiply the generating function for each associated progression by some eta quotient over the divisors of 40. We give the associated system of equations here (which may be checked by comparing to Theorem 45 of Radu's paper "An Algorithmic Approach to Ramanujan--Kolberg Identities").

The function sSys[N_,M_,r_,m_,j_,PM_] gives the associated system of equations that need to be solved. N here will be 40 (the level of our modular curve). M will be 20 (taking an eta quotient over the divisors of which our monomial in t, x, y, will be expressed), and r is the associated vector over the divisors of 20. Here m will be 4 and j will be the associated rem2 progression mod 4. PM will just be the set {rem2}. This is taken from key steps in the buildup of a given Ramanujan--Kolberg identity.

```
In[ ]:= w[r_] :=
  Sum[
   r[[d]],
   {d, 1, Length[r]}
  ]

In[ ]:= sigI[r_, N_] :=
  Sum[
   Divisors[N][[d]] × r[[d]],
   {d, 1, Length[Divisors[N]]}
  ]

In[ ]:= sig0[r_, N_] :=
  Sum[
   N/Divisors[N][[d]] r[[d]],
   {d, 1, Length[Divisors[N]]}
  ]

v[N_, M_, r_, m_, j2_, PM_] :=
  Sum[
   1/m (1 - m^2) (24 PM[[j]] + sigI[r, M]),
   {j, 1, Length[PM]}
  ]

In[ ]:= primeF[N_] :=
  Table[
   FactorInteger[N][[j]][[1]],
   {j, 1, Length[FactorInteger[N]]}
  ]
```

In[ ]:=
```
pp[N_, i_, d_] :=
 If[
  Mod[IntegerExponent[d, primeF[N][[i]]], 2] == 1,
  1,
  0
 ]

prodS[N_, M_, r_, m_, j2_, PM_] :=
 Product[
   (m Divisors[M][[i]])^(Abs[r[[i]]]), {i, 1, Length[Divisors[M]]}]^(Length[PM])

sSys[N_, M_, r_, m_, j2_, PM_] :=
 Join[
  {
   Sum[s[j], {j, 1, Length[Divisors[N]]}] == -Length[PM] × w[r],
   Sum[Divisors[N][[j]] × s[j], {j, 1, Length[Divisors[N]]}] + 24 mu11 ==
    Mod[-v[N, M, r, m, j2, PM] - Length[PM] m sigI[r, M], 24],
   Sum[N/Divisors[N][[j]] s[j], {j, Length[Divisors[N]]}] + 24 mu22 ==
    Mod[-Length[PM] (m N)/M sig0[r, M], 24]
  },
  Table[Sum[pp[N M, i, Divisors[N][[j]]] × s[j], {j, 1, Length[Divisors[N]]}] + 2 ep[i] ==
    pp[N M, i, prodS[N, M, r, m, j2, PM]], {i, 1, Length[primeF[N M]]}]
 ]
```

More good news: we will have to multiply the generating function for each associated progression by some eta quotient over the divisors of 40. But we can show that a single eta quotient will suffice for every single monomial that we have. To do this, we check whether the system of equations for our first monomial matches all of the others:

In[ ]:=
```
Table[
 sSys[
   40,
   20,
   Exponent[mLp01w[[1]], t] tv + Exponent[mLp01w[[1]], x] xv +
    Exponent[mLp01w[[1]], y] yv + {0, 0, 0, 2, -1, 0},
   4,
   rem2[Exponent[mLp01w[[1]], t], Exponent[mLp01w[[1]], x], Exponent[mLp01w[[1]], y]],
   {rem2[Exponent[mLp01w[[1]], t], Exponent[mLp01w[[1]], x], Exponent[mLp01w[[1]], y]]}
  ] == sSys[
   40,
   20,
   Exponent[mLp01w[[j1]], t] tv + Exponent[mLp01w[[j1]], x] xv +
    Exponent[mLp01w[[j1]], y] yv + {0, 0, 0, 2, -1, 0},
   4,
   rem2[Exponent[mLp01w[[j1]], t],
    Exponent[mLp01w[[j1]], x], Exponent[mLp01w[[j1]], y]],
   {rem2[Exponent[mLp01w[[j1]], t], Exponent[mLp01w[[j1]], x],
     Exponent[mLp01w[[j1]], y]]}
  ],
 {j1, 1, Length[mLp01w]}
]
```

Out[ ]=
```
{True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True}
```

One may solve this system of equations, together with the inequalities in Theorem 47 of Radu's paper "An Algorithmic Approach to Ramanujan--Kolberg Identities," to get an eta quotient which will induce nonnegative order at every cusp except [∞]. For the sake of simplicity we simply give one such eta quotient, given by the vector
(219, -93, -16, -45, 44, 18, 128, -256) indexed over the divisors of 40. We define the function

```
pStar[y_, N_, s_] :=
 1/24 Sum[
   s[[i]] GCD[Divisors[N][[i]], y[[2]][[1]]]^2 / Divisors[N][[i]],
   {i, 1, Length[Divisors[N]]}
  ]
```

again in keeping with Theorem 47 of Radu’s paper “An Algorithmic Approach to Ramanujan--Kolberg Identities.” We then put together the associated lower bound on the order of the associated generating function (keeping track of the progression 4n+2) for each monomial, with our normalizing eta quotient. Notice that the orders are all only at [∞]:

```
In[ ]:= Column[
  Table[
   Table[
    N[
                        40
     ─────────────────────────────────────── (
     GCD[(repMat[40][[l1]][[2]][[1]])^2, 40]
       p[repMat[40][[l1]], 20, Exponent[mLp01w[[j1]], t] tv + Exponent[mLp01w[[j1]], x] xv +
          Exponent[mLp01w[[j1]], y] yv + {0, 0, 0, 2, -1, 0}, 4] +
        pStar[repMat[40][[l1]], 40, {219, -93, -16, -45, 44, 18, 128, -256}]
      )
     ],
    {l1, 1, Length[repMat[40]]}
   ],
   {j1, 1, Length[mLp01w]}
  ]
 ]
```

Out[∘]=
{-306., 3., 4.5, 0.75, 18., 9.75, 12., 0.}
{-306.5, 2.5, 3.5, 0.5, 14., 9.5, 11.5, 24.}
{-305.75, 3.25, 5., 1., 20., 10., 12.5, 0.}
{-306.25, 2.75, 4., 0.75, 16., 9.75, 12., 24.}
{-306., 3., 4.5, 1., 18., 10., 12.5, 16.}
{-306.5, 2.5, 3.5, 0.75, 14., 9.75, 12., 40.}
{-306.25, 2.75, 4., 1., 16., 10., 12.5, 32.}
{-306.75, 2.25, 3., 0.75, 12., 9.75, 12., 56.}
{-306., 3., 4.5, 1.25, 18., 10.25, 13., 32.}
{-306.5, 2.5, 3.5, 1., 14., 10., 12.5, 56.}
{-306.25, 2.75, 4., 1.25, 16., 10.25, 13., 48.}
{-306.75, 2.25, 3., 1., 12., 10., 12.5, 72.}
{-306.5, 2.5, 3.5, 1.25, 14., 10.25, 13., 64.}
{-307., 2., 2.5, 1., 10., 10., 12.5, 88.}
{-306.25, 2.75, 4., 1.5, 16., 10.5, 13.5, 64.}
{-306.75, 2.25, 3., 1.25, 12., 10.25, 13., 88.}
{-306.5, 2.5, 3.5, 1.5, 14., 10.5, 13.5, 80.}
{-307., 2., 2.5, 1.25, 10., 10.25, 13., 104.}
{-306.75, 2.25, 3., 1.5, 12., 10.5, 13.5, 96.}
{-307.25, 1.75, 2., 1.25, 8., 10.25, 13., 120.}
{-306.5, 2.5, 3.5, 1.75, 14., 10.75, 14., 96.}
{-307., 2., 2.5, 1.5, 10., 10.5, 13.5, 120.}
{-306.75, 2.25, 3., 1.75, 12., 10.75, 14., 112.}
{-307.25, 1.75, 2., 1.5, 8., 10.5, 13.5, 136.}
{-307., 2., 2.5, 1.75, 10., 10.75, 14., 128.}
{-307.5, 1.5, 1.5, 1.5, 6., 10.5, 13.5, 152.}
{-306.75, 2.25, 3., 2., 12., 11., 14.5, 128.}
{-307.25, 1.75, 2., 1.75, 8., 10.75, 14., 152.}
{-307., 2., 2.5, 2., 10., 11., 14.5, 144.}
{-307.5, 1.5, 1.5, 1.75, 6., 10.75, 14., 168.}
{-307.25, 1.75, 2., 2., 8., 11., 14.5, 160.}
{-307.75, 1.25, 1., 1.75, 4., 10.75, 14., 184.}
{-307., 2., 2.5, 2.25, 10., 11.25, 15., 160.}
{-307.5, 1.5, 1.5, 2., 6., 11., 14.5, 184.}
{-307.25, 1.75, 2., 2.25, 8., 11.25, 15., 176.}
{-307.75, 1.25, 1., 2., 4., 11., 14.5, 200.}
{-307.5, 1.5, 1.5, 2.25, 6., 11.25, 15., 192.}
{-308., 1., 0.5, 2., 2., 11., 14.5, 216.}
{-307.25, 1.75, 2., 2.5, 8., 11.5, 15.5, 192.}
{-307.75, 1.25, 1., 2.25, 4., 11.25, 15., 216.}
{-307.5, 1.5, 1.5, 2.5, 6., 11.5, 15.5, 208.}
{-308., 1., 0.5, 2.25, 2., 11.25, 15., 232.}
{-307.75, 1.25, 1., 2.5, 4., 11.5, 15.5, 224.}
{-308.25, 0.75, 0., 2.25, 0., 11.25, 15., 248.}
{-307.5, 1.5, 1.5, 2.75, 6., 11.75, 16., 224.}
{-308., 1., 0.5, 2.5, 2., 11.5, 15.5, 248.}
{-307.75, 1.25, 1., 2.75, 4., 11.75, 16., 240.}
{-308.25, 0.75, 0., 2.5, 0., 11.5, 15.5, 264.}

We can therefore express each of these functions as elements enumerated by our algebra basis ab40. To properly give the principal parts and constants of each function, we define the following functions, as necessary in the construction of RaduRK:

```
In[ ]:= f1[s_, N_, M_, r_, m_, PM_] :=
  Product[
    Eta[Divisors[N][[i]]]^(s[[i]]),
    {i, 1, Length[Divisors[N]]}
   ] × Product[
    q^((24 PM[[i]] + sigI[r, M])/(24 m)),
    {i, 1, Length[PM]}
   ]
```

```
In[ ]:= fLHS0[s_, N_, M_, r_, m_, PM_] :=
  Module[
   {
    L =
     Series[Product[QPochhammer[q^(Divisors[M][[d]]), q^(Divisors[M][[d]])]^(r[[d]]),
       {d, 1, Length[Divisors[M]]}], {q, 0, m (pord[f1[s, N, M, r, m, PM]] + 1)}
     ],
    g =
     Normal[
      Series[
       f1[s, N, M, r, m, PM],
       {q, 0, 0}
      ]
     ]
   },
   1;
   If[
    s == {},
    1,
    Normal[
     Series[
      g Product[
        Sum[
         Coefficient[L, q, m n + PM[[i]]] q^n,
         {n, 0, pord[f1[s, N, M, r, m, PM]]}
        ],
        {i, 1, Length[PM]}
       ],
      {q, 0, 0}
     ]
    ]
   ]
  ]

In[ ]:= fLHS[s_, N_, M_, r_, m_, PM_] :=
  If[s == {} || s == {0}, "No Membership", fLHS0[s, N, M, r, m, PM]]
```

We now construct a table of the principal parts and constants of each function:

```
In[ ]:= fH =
  Table[
   {
    j1,
    fLHS[
     {219, -93, -16, -45, 44, 18, 128, -256},
     40,
     20,
     Exponent[mLp01w[[j1]], t] tv + Exponent[mLp01w[[j1]], x] xv +
      Exponent[mLp01w[[j1]], y] yv + {0, 0, 0, 2, -1, 0},
     4,
     {rem2[Exponent[mLp01w[[j1]], t],
       Exponent[mLp01w[[j1]], x], Exponent[mLp01w[[j1]], y]]}
    ]
   },
   {j1, 1, Length[mLp01w]}
  ];
```

Finally, we construct a table which contains the expression of each function as a member of the algebra basis (Notice that here, “t” represents our placeholder, in this case corresponding to ab40[[1]], not our Hauptmodul t):

This took approximately seven hours’ computation time (as did checking the similar case for 4n+3 below).

In[ ]:=
```
mW =
 Table[
  {
   j1,
   MW[
    ab40[[1]], ab40[[2]],
    fH[[j1]][[2]]
   ]
  },
  {j1, 1, Length[mLp01w]}
 ]
```

Out[ ]=
```
{{1,
  {196 159 429 230 833 773 869 868 419 475 239 575 503 198 607 639 501 078 528 000 000 000 000 000
     000 000 000 000 000 000 000 000 000 000 000 000 +
   2 550 072 580 000 839 060 308 289 453 178 114 481 541 581 899 313 514 020 864 000 000 000 000
     000 000 000 000 000 000 000 000 000 000 000 000 000 t + ⋯74⋯ + 5 367 006 440 t^76,
  ⋯2⋯, ⋯1⋯}}, ⋯46⋯, ⋯1⋯}
```

large output | show less | show more | show all | set size limit...

There is one more consideration: each of these monomials has a given integer coefficient, which we recorded in the table mLp01wC. We combine each element of mLp01wC with the function for the associated progression mod 4 of the monomial. We don't want to worry about dealing with the actual eta quotients of ab40, so we replace them with indeterminates z5, z6, z7.

Each of these functions has an eta quotient multiplied on to ensure modularity. However, we went to the trouble of making sure that these multipliers were all the same (the eta quotient given by the vector (219, -93, -16, -45, 44, 18, 128, -256) over the divisors of 40). As such, the multiplier will have no effect if the progression comes out to be zero.

In[ ]:=
```
Sum[
  mLp01wC[[j1]] × mW[[j1]][[2]].{1, z5, z6, z7},
  {j1, 1, Length[mLp01w]}
 ] // Together
```

Out[ ]=
```
0
```

Which it does!

$c\psi_{2,0}(n)$: Odd Index, Progression 4n+3

We work through the same steps as above for the progression 4n+3. We of course will modify our key progression and variation on the minN programs:

In[ ]:=
```
rem3[nt_, nx_, ny_] :=
 Mod[3 - (nt + nx + 2 ny), 4]
```

In[ ]:=
```
MN3[f_] :=
 minN[
  20,
  Exponent[f, t] tv + Exponent[f, x] xv + Exponent[f, y] yv + {0, 0, 0, 2, -1, 0},
  4,
  rem3[Exponent[f, t], Exponent[f, x], Exponent[f, y]]
 ]
```

We also of course have a different multiplier for our ultimate identities, which we will give here as V3:

In[ ]:=
```
V3 = {224, -107, -7, -48, 42, 26, 123, -254};
```

Otherwise, the steps are largely identical to those above.

In[ ]:=
```
Column[
 Table[
  {j, MN3[mLp01w[[j]]]},
  {j, 1, Length[mLp01w]}
 ]
]
```

Out[•]=

{1, 40}
{2, 40}
{3, 40}
{4, 40}
{5, 40}
{6, 40}
{7, 40}
{8, 40}
{9, 40}
{10, 40}
{11, 40}
{12, 40}
{13, 40}
{14, 40}
{15, 40}
{16, 40}
{17, 40}
{18, 40}
{19, 40}
{20, 40}
{21, 40}
{22, 40}
{23, 40}
{24, 40}
{25, 40}
{26, 40}
{27, 40}
{28, 40}
{29, 40}
{30, 40}
{31, 40}
{32, 40}
{33, 40}
{34, 40}
{35, 40}
{36, 40}
{37, 40}
{38, 40}
{39, 40}
{40, 40}
{41, 40}
{42, 40}
{43, 40}
{44, 40}
{45, 40}
{46, 40}
{47, 40}
{48, 40}

In[ ]:=
```
Pmr3[f_] :=
 Pmr[
  40,
  20,
  Exponent[f, t] tv + Exponent[f, x] xv + Exponent[f, y] yv + {0, 0, 0, 2, -1, 0},
  4,
  rem3[Exponent[f, t], Exponent[f, x], Exponent[f, y]]
 ]
```

In[ ]:=
```
Column[
 Table[
  {j, Length[Pmr3[mLp01w[[j]]]]},
  {j, 1, Length[mLp01w]}
 ]
]
```

Out[•]=

```
{1, 1}
{2, 1}
{3, 1}
{4, 1}
{5, 1}
{6, 1}
{7, 1}
{8, 1}
{9, 1}
{10, 1}
{11, 1}
{12, 1}
{13, 1}
{14, 1}
{15, 1}
{16, 1}
{17, 1}
{18, 1}
{19, 1}
{20, 1}
{21, 1}
{22, 1}
{23, 1}
{24, 1}
{25, 1}
{26, 1}
{27, 1}
{28, 1}
{29, 1}
{30, 1}
{31, 1}
{32, 1}
{33, 1}
{34, 1}
{35, 1}
{36, 1}
{37, 1}
{38, 1}
{39, 1}
{40, 1}
{41, 1}
{42, 1}
{43, 1}
{44, 1}
{45, 1}
{46, 1}
{47, 1}
{48, 1}
```

In[ ]:=
```
Table[
 sSys[
   40,
   20,
   Exponent[mLp01w[[1]], t] tv + Exponent[mLp01w[[1]], x] xv +
    Exponent[mLp01w[[1]], y] yv + {0, 0, 0, 2, -1, 0},
   4,
   rem3[Exponent[mLp01w[[1]], t], Exponent[mLp01w[[1]], x], Exponent[mLp01w[[1]], y]],
   {rem3[Exponent[mLp01w[[1]], t], Exponent[mLp01w[[1]], x], Exponent[mLp01w[[1]], y]]}
  ] == sSys[
   40,
   20,
   Exponent[mLp01w[[j1]], t] tv + Exponent[mLp01w[[j1]], x] xv +
    Exponent[mLp01w[[j1]], y] yv + {0, 0, 0, 2, -1, 0},
   4,
   rem3[Exponent[mLp01w[[j1]], t],
    Exponent[mLp01w[[j1]], x], Exponent[mLp01w[[j1]], y]],
   {rem3[Exponent[mLp01w[[j1]], t], Exponent[mLp01w[[j1]], x],
     Exponent[mLp01w[[j1]], y]]}
  ],
 {j1, 1, Length[mLp01w]}
]
```

Out[ ]=
```
{True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True}
```

In[ ]:=
```
Column[
 Table[
  Table[
   N[
    40/GCD[(repMat[40][[l1]][[2]][[1]])^2, 40] (
      p[repMat[40][[l1]], 20, Exponent[mLp01w[[j1]], t] tv +
         Exponent[mLp01w[[j1]], x] xv + Exponent[mLp01w[[j1]], y] yv +
         {0, 0, 0, 2, -1, 0}, 4] + pStar[repMat[40][[l1]], 40, V3]
      )
    ],
   {l1, 1, Length[repMat[40]]}
   ],
  {j1, 1, Length[mLp01w]}
  ]
 ]
```

Out[•]=

{-304.25, 2.75, 7., 0.5, 18., 11.5, 6.5, 0.}
{-304.75, 2.25, 6., 0.25, 14., 11.25, 6., 24.}
{-304., 3., 7.5, 0.75, 20., 11.75, 7., 0.}
{-304.5, 2.5, 6.5, 0.5, 16., 11.5, 6.5, 24.}
{-304.25, 2.75, 7., 0.75, 18., 11.75, 7., 16.}
{-304.75, 2.25, 6., 0.5, 14., 11.5, 6.5, 40.}
{-304.5, 2.5, 6.5, 0.75, 16., 11.75, 7., 32.}
{-305., 2., 5.5, 0.5, 12., 11.5, 6.5, 56.}
{-304.25, 2.75, 7., 1., 18., 12., 7.5, 32.}
{-304.75, 2.25, 6., 0.75, 14., 11.75, 7., 56.}
{-304.5, 2.5, 6.5, 1., 16., 12., 7.5, 48.}
{-305., 2., 5.5, 0.75, 12., 11.75, 7., 72.}
{-304.75, 2.25, 6., 1., 14., 12., 7.5, 64.}
{-305.25, 1.75, 5., 0.75, 10., 11.75, 7., 88.}
{-304.5, 2.5, 6.5, 1.25, 16., 12.25, 8., 64.}
{-305., 2., 5.5, 1., 12., 12., 7.5, 88.}
{-304.75, 2.25, 6., 1.25, 14., 12.25, 8., 80.}
{-305.25, 1.75, 5., 1., 10., 12., 7.5, 104.}
{-305., 2., 5.5, 1.25, 12., 12.25, 8., 96.}
{-305.5, 1.5, 4.5, 1., 8., 12., 7.5, 120.}
{-304.75, 2.25, 6., 1.5, 14., 12.5, 8.5, 96.}
{-305.25, 1.75, 5., 1.25, 10., 12.25, 8., 120.}
{-305., 2., 5.5, 1.5, 12., 12.5, 8.5, 112.}
{-305.5, 1.5, 4.5, 1.25, 8., 12.25, 8., 136.}
{-305.25, 1.75, 5., 1.5, 10., 12.5, 8.5, 128.}
{-305.75, 1.25, 4., 1.25, 6., 12.25, 8., 152.}
{-305., 2., 5.5, 1.75, 12., 12.75, 9., 128.}
{-305.5, 1.5, 4.5, 1.5, 8., 12.5, 8.5, 152.}
{-305.25, 1.75, 5., 1.75, 10., 12.75, 9., 144.}
{-305.75, 1.25, 4., 1.5, 6., 12.5, 8.5, 168.}
{-305.5, 1.5, 4.5, 1.75, 8., 12.75, 9., 160.}
{-306., 1., 3.5, 1.5, 4., 12.5, 8.5, 184.}
{-305.25, 1.75, 5., 2., 10., 13., 9.5, 160.}
{-305.75, 1.25, 4., 1.75, 6., 12.75, 9., 184.}
{-305.5, 1.5, 4.5, 2., 8., 13., 9.5, 176.}
{-306., 1., 3.5, 1.75, 4., 12.75, 9., 200.}
{-305.75, 1.25, 4., 2., 6., 13., 9.5, 192.}
{-306.25, 0.75, 3., 1.75, 2., 12.75, 9., 216.}
{-305.5, 1.5, 4.5, 2.25, 8., 13.25, 10., 192.}
{-306., 1., 3.5, 2., 4., 13., 9.5, 216.}
{-305.75, 1.25, 4., 2.25, 6., 13.25, 10., 208.}
{-306.25, 0.75, 3., 2., 2., 13., 9.5, 232.}
{-306., 1., 3.5, 2.25, 4., 13.25, 10., 224.}
{-306.5, 0.5, 2.5, 2., 0., 13., 9.5, 248.}
{-305.75, 1.25, 4., 2.5, 6., 13.5, 10.5, 224.}
{-306.25, 0.75, 3., 2.25, 2., 13.25, 10., 248.}
{-306., 1., 3.5, 2.5, 4., 13.5, 10.5, 240.}
{-306.5, 0.5, 2.5, 2.25, 0., 13.25, 10., 264.}

In[ ]:=
```
fH3 =
  Table[
   {
    j1,
    fLHS[
     V3,
     40,
     20,
     Exponent[mLp01w[[j1]], t] tv + Exponent[mLp01w[[j1]], x] xv +
      Exponent[mLp01w[[j1]], y] yv + {0, 0, 0, 2, -1, 0},
     4,
     {rem3[Exponent[mLp01w[[j1]], t],
       Exponent[mLp01w[[j1]], x], Exponent[mLp01w[[j1]], y]]}
    ]
   },
   {j1, 1, Length[mLp01w]}
  ];
```

In[ ]:=
```
mW3 =
 Table[
  {
   j1,
   MW[
    ab40[[1]], ab40[[2]],
    fH3[[j1]][[2]]
   ]
  },
  {j1, 1, Length[mLp01w]}
 ]
```

Out[ ]=
```
{{1,
  {19 615 942 923 083 377 386 986 841 947 523 957 550 319 860 763 950 107 852 800 000 000 000 000
     000 000 000 000 000 000 000 000 000 000 000 000 +
   253 045 663 707 775 568 292 130 261 123 059 052 399 126 203 854 956 391 301 120 000 000 000 000
     000 000 000 000 000 000 000 000 000 000 000 000 t + ... 74 ... + 53 t^76,
   ... 1 ..., ... 1 ..., ... 1 ...}}, ... 47 ...}
```

large output | show less | show more | show all | set size limit...

In[ ]:=
```
Sum[
  mLp01wC[[j1]] × mW3[[j1]][[2]].{1, z5, z6, z7},
  {j1, 1, Length[mLp01w]}
 ] // Together
```

Out[ ]= 0