Congruences for the coefficients of the Gordon and McIntosh mock theta function $\xi(q)$

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Abstract Recently Gordon and McIntosh introduced the third order mock theta function $\xi(q)$ defined by

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n^2 - 6n + 1}}{(q; q^6)_n (q^5; q^6)_n}.$$

Our goal in this paper is to study arithmetic properties of the coefficients of this function. We present a number of such properties, including several infinite families of Ramanujan–like congruences.

Keywords congruence \cdot generating function \cdot mock theta function

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1 Introduction

In his last letter to Hardy in 1920, Ramanujan introduced the notion of a mock theta function. He listed 17 such functions having orders 3, 5, and 7. Since then, other mock theta functions have been found. Gordon and McIntosh [7], for example, introduced many additional such functions, including the following of order 3:

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n^2 - 6n + 1}}{(q; q^6)_n (q^5; q^6)_n},\tag{1}$$

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where we use the standard q-series notation:

$$(a;q)_0 = 1,(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \forall n \ge 1,(a;q)_\infty = \lim_{n \to \infty} (a;q)_n, |q| < 1.$$

Arithmetic properties of the coefficients of mock theta functions have received a great deal of attention. For instance, Zhang and Shi [15] recently proved seven congruences satisfied by the coefficients of the mock theta function $\beta(q)$ introduced by McIntosh. In a recent paper, Brietzke, da Silva, and Sellers [5] found a number of arithmetic properties satisfied by the coefficients of the mock theta function $V_0(q)$, introduced by Gordon and McIntosh [6]. Andrews et al. [2] prove a number of congruences for the partition functions $p_{\omega}(n)$ and $p_{\nu}(n)$, introduced in [1], associated with the third order mock theta functions $\omega(q)$ and $\nu(q)$, where $\omega(q)$ is defined below and

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$$

In a subsequent paper, Wang [14] presented some additional congruences for both $p_{\omega}(n)$ and $p_{\nu}(n)$.

This paper is devoted to exploring arithmetic properties of the coefficients $p_{\xi}(n)$ defined by

$$\sum_{n=0}^{\infty} p_{\xi}(n)q^n = \xi(q).$$
(2)

It is clear from (1) that $p_{\xi}(n)$ is even for all $n \geq 1$. In Sections 4 and 5, we present other arithmetic properties of $p_{\xi}(n)$, including some infinite families of congruences.

2 Preliminaries

McIntosh [12, Theorem 3] proved a number of mock theta conjectures, including

$$\omega(q) = g_3(q, q^2) \quad \text{and} \tag{3}$$

$$\xi(q) = q^2 g_3(q^3, q^6) + \frac{(q^2; q^2)_\infty^4}{(q; q)_\infty^2 (q^6; q^6)_\infty},\tag{4}$$

where

$$g_3(a,q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(a;q)_{n+1} (a^{-1}q;q)_{n+1}}$$

and $\omega(q)$ is the third order mock theta functions given by

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}.$$

It follows from (1), (3), and (4) that

$$\xi(q) = q^2 \omega(q^3) + \frac{(q^2; q^2)_{\infty}^4}{(q; q)_{\infty}^2 (q^6; q^6)_{\infty}}.$$
(5)

Throughout the remainder of this paper, we define

$$f_k := (q^k; q^k)_{\infty}$$

in order to shorten the notation. Combining (5) and (2), we have

$$\sum_{n=0}^{\infty} p_{\xi}(n)q^n = q^2\omega(q^3) + \frac{f_2^4}{f_1^2 f_6}.$$
(6)

We recall Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{ for } |ab| < 1,$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_2^2 f_2^2}, \text{ and}$$
(7)

$$\psi(q) := f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{\epsilon}.$$
(8)

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{J_2}{f_1}.$$
(8)

The function $\phi(q)$ satisfies many identities, including (see [3, (22.4)])

$$\phi(-q) = \frac{f_1^2}{f_2}.$$
(9)

In some of the proofs, we employ the classical Jacobi's identity (see [4, Theorem 1.3.9])

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$
 (10)

We note the following identities which will be used below.

Lemma 1 The following 2-dissection identities hold.

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8},\tag{11}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}, \tag{12}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}},\tag{13}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}},\tag{14}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}},\tag{15}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2} \tag{16}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7},\tag{17}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}$$
(18)

Proof By Entry 25 (i), (ii), (v), and (vi) in [3, p. 40], we have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{19}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2.$$
(20)

Using (7) and (8) we can rewrite (19) in the form

$$\frac{f_2^5}{f_1^2f_4^2} = \frac{f_8^5}{f_4^2f_{16}^2} + 2q\frac{f_{16}^2}{f_8},$$

from which we obtain (11) after multiplying both sides by $\frac{f_4^2}{f_2^5}$. Identity (12) can be easily deduced from (11) using the procedure described in Section 30.10 of [9].

By (7) and (8) we can rewrite (20) in the form

$$\frac{f_2^{10}}{f_1^4 f_4^4} = \frac{f_4^{10}}{f_2^4 f_8^4} + 4q \frac{f_8^4}{f_4^2},$$

from which we obtain (13).

Identities (14), (15), and (18) are equations (30.10.3), (30.9.9), and (30.12.3) of [9], respectively. Finally, for proofs of (16) and (17) see [13, Lemma 4].

The next lemma exhibits the 3-dissections of $\psi(q)$ and $1/\phi(-q)$.

Lemma 2 We have

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9},\tag{21}$$

$$\frac{1}{\phi(-q)} = \frac{f_6^4 f_9^6}{f_8^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_6^3}.$$
(22)

Proof Identity (21) is equation (14.3.3) of [9]. A proof of (22) can be seen in [11].

3 Dissections for $p_{\xi}(n)$

This section is devoted to proving the 2-, 3-, and 4-dissections of (2). We begin with the 2-dissection.

Theorem 1 We have

$$2\sum_{n=0}^{\infty} p_{\xi}(2n+1)q^{n+1} = \frac{f_6^6 f_{12}}{f_3^4 f_{24}^2} - f(q^{12}) + 4q \frac{f_2^2 f_8^2}{f_1 f_3 f_4}, \quad and \tag{23}$$

$$\sum_{n=0}^{\infty} p_{\xi}(2n)q^n = q \frac{f_6^8 f_{24}^2}{f_3^4 f_{12}^5} - q^4 \omega(-q^6) + \frac{f_4^5}{f_1 f_3 f_8^2}.$$
 (24)

Proof We start with equation (4) of [2]:

$$f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) = F(q),$$

where f(q) is the mock theta function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

and

$$F(q) = \frac{\phi(q)\phi(q^2)^2}{f_4^2} = \frac{f_2 f_4^4}{f_1^2 f_8^4}$$

Thus,

$$f(q^{24}) + 2q^3\omega(q^3) + 2q^9\omega(-q^{12}) = F(q^3).$$

Using (5), it follows that

$$2\sum_{n=0}^{\infty} p_{\xi}(n)q^{n+1} = F(q^3) - f(q^{24}) - 2q^9\omega(-q^{12}) + 2q\frac{f_2^4}{f_1^2f_6}.$$
 (25)

By (11), we have

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$$F(q^3) = \frac{f_{12}^6 f_{24}}{f_6^4 f_{48}^2} + 2q^3 \frac{f_{12}^8 f_{48}^2}{f_6^4 f_{24}^5},$$

which along with (11) allows us to rewrite (25) as

$$2\sum_{n=0}^{\infty} p_{\xi}(n)q^{n+1} = \frac{f_{12}^{6}f_{24}}{f_{6}^{4}f_{48}^{2}} + 2q^{3}\frac{f_{12}^{8}f_{48}^{2}}{f_{6}^{4}f_{24}^{5}} - f(q^{24}) - 2q^{9}\omega(-q^{12}) + 2q\frac{f_{8}^{5}}{f_{2}f_{6}f_{16}^{2}} + 4q^{2}\frac{f_{4}^{2}f_{16}^{2}}{f_{2}f_{6}f_{8}}.$$

Thus,

$$2\sum_{n=0}^{\infty} p_{\xi}(2n+1)q^{2n+2} = \frac{f_{12}^6 f_{24}}{f_6^4 f_{48}^2} - f(q^{24}) + 4q^2 \frac{f_4^2 f_{16}^2}{f_2 f_6 f_8}, \quad \text{and} \qquad (26)$$

$$\sum_{n=0}^{\infty} p_{\xi}(2n)q^{2n+1} = q^3 \frac{f_{12}^8 f_{48}^2}{f_6^4 f_{24}^5} - q^9 \omega(-q^{12}) + q \frac{f_8^5}{f_2 f_6 f_{16}^2}.$$
 (27)

Dividing (27) by q and replacing q^2 by q in the resulting identity and in (26), we obtain (23) and (24).

The next theorem exhibits the 3-dissection of (2).

Theorem 2 We have

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^n = \frac{f_2 f_3^4}{f_1^2 f_6^2},$$
(28)

$$\sum_{n=0}^{\infty} p_{\xi}(3n+1)q^n = 2\frac{f_3f_6}{f_1}, \quad and \tag{29}$$

$$\sum_{n=0}^{\infty} p_{\xi}(3n+2)q^n = \omega(q) + \frac{f_6^4}{f_2 f_3^2}.$$
(30)

Proof In view of (8), we rewrite (6) as

$$\sum_{n=0}^{\infty} p_{\xi}(n)q^n = q^2\omega(q^3) + \frac{\psi(q)^2}{f_6}.$$

Using (21), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(n)q^n = q^2\omega(q^3) + \frac{f_6f_9^4}{f_3^2f_{18}^2} + 2q\frac{f_9f_{18}}{f_3} + q^2\frac{f_{18}^4}{f_6f_9^2}.$$
 (31)

Extracting the terms of the form q^{3n+r} on both sides of (31), for $r \in \{0, 1, 2\}$, dividing both sides of the resulting identity by q^r and then replacing q^3 by q, we obtain the desired results.

We close this section with the 4-dissection of (2).

Theorem 3 We have

$$\sum_{n=0}^{\infty} p_{\xi}(4n)q^n = 4q^2 \frac{f_{12}^6}{f_3^2 f_6^3} - q^2 \omega(-q^3) + \frac{f_2^4 f_6^5}{f_1^2 f_3^4 f_{12}^2},$$
 (32)

$$\sum_{n=0}^{\infty} p_{\xi}(4n+1)q^n = 2q \frac{f_6^3 f_{12}^2}{f_3^4} + 2 \frac{f_4^4 f_5^5}{f_2^2 f_3^4 f_{12}^2},$$
(33)

$$\sum_{n=0}^{\infty} p_{\xi}(4n+2)q^n = \frac{f_6^9}{f_3^6 f_{12}^2} + \frac{f_2^{10} f_{12}^2}{f_1^4 f_3^2 f_4^4 f_6}, \quad and \tag{34}$$

$$2\sum_{n=0}^{\infty} p_{\xi}(4n+3)q^{n+1} = \frac{f_6^{15}}{f_3^8 f_{12}^6} - f(q^6) + 4q \frac{f_2^4 f_{12}^2}{f_1^2 f_3^2 f_6^2}.$$
 (35)

Proof In order to prove (32), we use (13) and (18) to obtain the even part of (24), which is given by

$$\sum_{n=0}^{\infty} p_{\xi}(4n)q^{2n} = 4q^4 \frac{f_{24}^6}{f_6^2 f_{12}^3} - q^4 \omega(-q^6) + \frac{f_4^4 f_{12}^5}{f_2^2 f_6^4 f_{24}^2}.$$

Replacing q^2 by q we obtain (32).

Using (13) and (18) we can extract the odd part of (23):

$$2\sum_{n=0}^{\infty} p_{\xi}(4n+1)q^{2n+1} = 4q^3 \frac{f_{12}^3 f_{24}^2}{f_6^4} + 4q \frac{f_8^4 f_{12}^5}{f_4^2 f_6^4 f_{24}^2}$$

After simplifications we arrive at (33).

Next, extracting the odd part of (24) with the help of (13) and (18) yields

$$\sum_{n=0}^{\infty} p_{\xi}(4n+2)q^{2n+1} = q \frac{f_{12}^9}{f_6^6 f_{24}^2} + q \frac{f_{10}^{40} f_{24}^2}{f_2^4 f_6^2 f_8^4 f_{12}^4},$$

which, after simplifications, gives us (34).

In order to obtain (35), we use (13) and (18) in (23) to extract its even part:

$$2\sum_{n=0}^{\infty} p_{\xi}(4n+3)q^{2n+2} = \frac{f_{12}^{15}}{f_6^8 f_{24}^6} - f(q^{12}) + 4q^2 \frac{f_4^4 f_{24}^2}{f_2^2 f_6^2 f_{12}}$$

Replacing q^2 by q in this identity, we obtain (35).

4 Arithmetic properties of $p_{\xi}(n)$

Our first observation provides a characterization of $p_{\xi}(3n) \pmod{4}$.

Theorem 4 For all $n \ge 0$, we have

$$p_{\xi}(3n) \equiv \begin{cases} 1 \pmod{4}, & \text{if } n = 0, \\ 2 \pmod{4}, & \text{if } n \text{ is a square,} \\ 0 \pmod{4}, & \text{otherwise.} \end{cases}$$

Proof By (28), using (9) and the fact that $f_k^4 \equiv f_{2k}^2 \pmod{4}$ for all $k \ge 1$, it follows that

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^n = \frac{f_2 f_3^4}{f_1^2 f_6^2} \equiv \frac{f_2}{f_1^2} = \frac{f_1^2 f_2}{f_1^4} \equiv \frac{f_1^2}{f_2} = \phi(-q) \pmod{4}.$$

By (7), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^n \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2} \pmod{4},$$

which completes the proof.

Theorem 4 yields an infinite family of Ramanujan–like congruences modulo 4.

Corollary 1 For all primes $p \ge 3$ and all $n \ge 0$, we have

$$p_{\xi}(3(pn+r)) \equiv 0 \pmod{4},$$

if r is a quadratic nonresidue modulo p.

Proof If $pn + r = k^2$, then $r \equiv k^2 \pmod{p}$, which contradicts the fact that r is a quadratic nonresidue modulo p.

Since gcd(3, p) = 1, among the p - 1 residues modulo p, we have $\frac{p-1}{2}$ residues r for which r is a quadratic nonresidue modulo p. Thus, for instance, the above corollary yields the following congruences:

$$p_{\xi}(9n+6) \equiv 0 \pmod{4},$$

$$p_{\xi}(15n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{6,9\},$$

$$p_{\xi}(21n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{9,15,18\},$$

$$p_{\xi}(33n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{6,18,21,24,30\}.$$

Theorem 5 For all $n \ge 0$, we have

$$p_{\xi}(3n+1) \equiv \begin{cases} 2 \pmod{4}, & \text{if } 3n+1 \text{ is a square,} \\ 0 \pmod{4}, & \text{otherwise.} \end{cases}$$

Proof From Theorem 2,

$$\sum_{n=0}^{\infty} p_{\xi}(3n+1)q^n = 2\frac{f_3f_6}{f_1}.$$
(36)

So we only need to consider the parity of

$$\frac{f_3f_6}{f_1}.$$

Note that

$$\frac{f_3f_6}{f_1} \equiv \frac{f_3^3}{f_1} = \sum_{n=0}^{\infty} a_3(n)q^n \pmod{2},$$

where $a_3(n)$ is the number of 3-core partitions of n (see [10, Theorem 1]). Thanks to [8, Theorem 7], we know that

$$a_3(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } 3n+1 \text{ is a square,} \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

This completes the proof.

Theorem 5 yields an infinite family of congruences modulo 4.

Corollary 2 For all primes p > 3 and all $n \ge 0$, we have

 $p_{\xi}(3(pn+r)+1) \equiv 0 \pmod{4},$

if 3r + 1 is a quadratic nonresidue modulo p.

Proof If $3(pn + r) + 1 = k^2$, then $3r + 1 \equiv k^2 \pmod{p}$, which would be a contradiction with 3r + 1 being a quadratic nonresidue modulo p.

For example, the following congruences hold for all $n \ge 0$:

 $p_{\xi}(15n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{7,13\}, \\ p_{\xi}(21n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{10,13,19\}, \\ p_{\xi}(33n+k) \equiv 0 \pmod{4}, \text{ for } k \in \{7,10,13,19,28\}.$

We next turn our attention to the arithmetic progression 4n + 2 to yield an additional infinite family of congruences.

Theorem 6 For all $n \ge 0$, we have

$$p_{\xi}(4n+2) \equiv \begin{cases} 2 \pmod{4}, & \text{if } n = 6k(3k \pm 1), \\ 0 \pmod{4}, & \text{otherwise.} \end{cases}$$

Proof From (34), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(4n+2)q^n \equiv \frac{f_6^7}{f_3^2 f_{12}^2} + \frac{f_{12}^2}{f_3^2 f_6} \equiv 2\frac{f_6^3}{f_3^2} \equiv 2f_6^2 \equiv 2f_{12} \pmod{4}.$$
(37)

Using Euler's identity (see [9, Eq. (1.6.1)])

$$f_1 = \sum_{n = -\infty}^{\infty} (-1)^n q^{n(3n-1)/2},$$
(38)

we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(4n+2)q^n \equiv 2\sum_{n=-\infty}^{\infty} (-1)^n q^{6n(3n-1)} \pmod{4}$$

which concludes the proof.

Theorem 6 yields an infinite family of congruences modulo 4.

Corollary 3 Let p > 3 be a prime and r an integer such that 2r + 1 is a quadratic nonresidue modulo p. Then, for all $n \ge 0$,

$$p_{\xi}(4(pn+r)+2) \equiv 0 \pmod{4}.$$

Proof If $pn+r = 6k(3k\pm 1)$, then $r \equiv 18k^2\pm 6k \pmod{p}$. Thus, $2r+1 \equiv (6k\pm 1)^2 \pmod{p}$, which contradicts the fact that 2r+1 is a quadratic nonresidue modulo p.

Thanks to Corollary 3, the following example congruences hold for all $n \geq 0$:

 $p_{\xi}(20n+j) \equiv 0 \pmod{4}, \text{ for } j \in \{6,14\},$ $p_{\xi}(28n+j) \equiv 0 \pmod{4}, \text{ for } j \in \{6,10,26\},$ $p_{\xi}(44n+j) \equiv 0 \pmod{4}, \text{ for } j \in \{14,26,34,38,42\},$ $p_{\xi}(52n+j) \equiv 0 \pmod{4}, \text{ for } j \in \{10,14,22,30,38,42\}.$

We now provide a mod 8 characterization for $p_{\xi}(3n).$

Theorem 7 For all $n \ge 0$, we have

$$p_{\xi}(3n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n = 0, \\ 6(-1)^k \pmod{8}, & \text{if } n = k^2, \\ 4 \pmod{8}, & \text{if } n = 2k^2, \ n = 3k^2, \ or \ n = 6k^2, \\ 0 \pmod{8}, & \text{otherwise.} \end{cases}$$

Proof By (28), using (7) and (9), we have

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^{n} = \frac{f_{1}^{6}f_{2}f_{3}^{4}}{f_{1}^{8}f_{6}^{2}} \equiv \left(\frac{f_{1}^{2}}{f_{2}}\right)^{3} \left(\frac{f_{3}^{2}}{f_{6}}\right)^{2} \equiv \phi(-q)^{3}\phi(-q^{3})^{2}$$
$$\equiv \left(\sum_{n=-\infty}^{\infty} (-1)^{n}q^{n^{2}}\right)^{3} \left(\sum_{n=-\infty}^{\infty} (-1)^{n}q^{3n^{2}}\right)^{2}$$
$$\equiv \left(1+2\sum_{n=1}^{\infty} (-1)^{n}q^{n^{2}}\right)^{3} \left(1+2\sum_{n=1}^{\infty} (-1)^{n}q^{3n^{2}}\right)^{2} \pmod{8}$$

which yields

$$\sum_{n=0}^{\infty} p_{\xi}(3n)q^{n} \equiv 1 + 6\sum_{n=1}^{\infty} (-1)^{n} q^{n^{2}} + 4\left(\sum_{n=1}^{\infty} (-1)^{n} q^{n^{2}}\right)^{2} + 4\sum_{n=1}^{\infty} (-1)^{n} q^{3n^{2}} + 4\left(\sum_{n=1}^{\infty} (-1)^{n} q^{3n^{2}}\right)^{2} \pmod{8}.$$

Since

$$\left(\sum_{n=1}^{\infty} (-1)^n q^{n^2}\right)^2 \equiv \sum_{n=1}^{\infty} q^{2n^2} \pmod{2},$$

we have

$$\sum_{n=1}^{\infty} (-1)^n q^{3n^2} \right)^2 \equiv \sum_{n=1}^{\infty} q^{6n^2} \pmod{2}.$$

Therefore

$$\begin{split} \sum_{n=0}^{\infty} p_{\xi}(3n) q^n &\equiv 1 + 6 \sum_{n=1}^{\infty} (-1)^n q^{n^2} + 4 \sum_{n=1}^{\infty} q^{2n^2} \\ &+ 4 \sum_{n=1}^{\infty} (-1)^n q^{3n^2} + 4 \sum_{n=1}^{\infty} q^{6n^2} \pmod{8}, \end{split}$$

which completes the proof.

As with the prior results, Theorem 7 provides an effective way to yield an infinite family of congruences modulo 8.

Corollary 4 Let p be a prime such that $p \equiv \pm 1 \pmod{24}$. Then,

$$p_{\xi}(3(pn+r)) \equiv 0 \pmod{8},$$

if r is a quadratic nonresidue modulo p.

Proof Since $p \equiv \pm 1 \pmod{8}$ and $p \equiv \pm 1 \pmod{12}$, it follows that 2 and 3 are quadratic residues modulo p. Thus, r, 2r, 3r, and 6r are quadratic nonresidues modulo p. Indeed, according to the properties of Legendre's symbol, for $j \in \{1, 2, 3, 6\}$, we have

$$\left(\frac{jr}{p}\right) = \left(\frac{j}{p}\right)\left(\frac{r}{p}\right) = \left(\frac{r}{p}\right) = -1.$$

It follows that we cannot have $3(pn + r) = jk^2$, for some $k \in \mathbb{N}$ and $j \in \{1, 2, 3, 6\}$. In fact, $3(pn+r) = jk^2$ would imply $3(pn+r) \equiv 3r \equiv jk^2 \pmod{p}$. However, for j = 1, 2, 3, 6, this would imply that 3r, 6r, r, or 2r, respectively, is a quadratic residue modulo p, which would be a contradiction since 2, 3, and 6 are quadratic residues modulo p. The result follows from Theorem 7.

As an example, we note that, for p = 23 and all $n \ge 0$, we have

 $p_{\xi}(69n+k) \equiv 0 \pmod{8}$, for $k \in \{15, 21, 30, 33, 42, 45, 51, 57, 60, 63, 66\}.$

Theorem 8 For all $n \ge 0$, we have

 $p_{\xi}(12n+4) \equiv p_{\xi}(3n+1) \pmod{8}.$

Proof Initially we use (14) to extract the odd part on both sides of (29). The resulting identity is

$$\sum_{n=0}^{\infty} p_{\xi}(6n+4)q^n = 2\frac{f_3^2 f_4^2 f_{24}}{f_1^2 f_8 f_{12}}.$$
(39)

Using (15) in (39), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^n = 2\frac{f_2^6 f_3 f_6}{f_1^5 f_4^2} = 2\frac{f_1^3 f_2^6 f_3 f_6}{f_1^8 f_4^2} \equiv 2\frac{f_3 f_6}{f_1} \pmod{8}.$$

The result follows using (29).

Now we present complete characterizations of $p_{\xi}(48n+4)$ and $p_{\xi}(12n+1)$ modulo 8.

Theorem 9 For all $n \ge 0$, we have

$$p_{\xi}(48n+4) \equiv p_{\xi}(12n+1) \equiv \begin{cases} 2(-1)^k \pmod{8}, & \text{if } n = k(3k \pm 1), \\ 0 \pmod{8}, & \text{otherwise.} \end{cases}$$

Proof The first congruence follows directly from Theorem 8. Replacing (14) in (29), we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(3n+1)q^n = 2\frac{f_4f_6^2f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + 2q\frac{f_6^2f_8^2f_{48}}{f_2^2f_{16}f_{24}}.$$

Extracting the terms of the form q^{2n} , we have

$$\sum_{n=0}^{\infty} p_{\xi}(6n+1)q^{2n} = 2\frac{f_4 f_6^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}}$$

which, after replacing q^2 by q, yields

$$\sum_{n=0}^{\infty} p_{\xi}(6n+1)q^n = 2\frac{f_2 f_3^2 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}}.$$
(40)

Now we use (15) to obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+1)q^n = 2\frac{f_2^3 f_6^4}{f_1^4 f_{12}^2}$$
$$\equiv 2f_2 \equiv 2\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)} \pmod{8}, \quad (by \ (38))$$

which completes the proof.

Theorem 9 also provides an effective way to yield an infinite family of congruences modulo 8.

Corollary 5 For all primes p > 3 and all $n \ge 0$, we have

$$p_{\xi}(48(pn+r)+4) \equiv p_{\xi}(12(pn+r)+1) \equiv 0 \pmod{8},$$

if 12r + 1 is a quadratic nonresidue modulo p.

Proof Let p > 3 be a prime and 12r + 1 a quadratic nonresidue modulo p. If $pn + r = k(3k \pm 1)$, then $r \equiv 3k^2 \pm k \pmod{p}$, which implies that $12r + 1 \equiv (6k \pm 1)^2 \pmod{p}$, a contradiction. The result follows from Theorem 9.

5 Additional congruences

In this section, we prove several additional Ramanujan–like congruences that are not included in the results of the previous section.

Theorem 10 For all $n \ge 0$, we have

$$p_{\xi}(24n+19) \equiv 0 \pmod{3},$$
 (41)

$$p_{\xi}(27n+18) \equiv 0 \pmod{3}, \text{ and} \tag{42}$$

$$p_{\xi}(72n+51) \equiv 0 \pmod{3}.$$
 (43)

Proof Using (15) we can now 2-dissect (40) to obtain

$$\sum_{n=0}^{\infty} p_{\xi}(6n+1)q^n = 2\frac{f_4^3 f_{12}^4}{f_2^4 f_{24}^2} + 4q \frac{f_6 f_8^2 f_{12}}{f_2^3},$$

from which we have

$$\sum_{n=0}^{\infty} p_{\xi}(12n+7)q^{2n+1} = 4q \frac{f_6 f_8^2 f_{12}}{f_2^3}.$$

Now, dividing both sides of the above expression by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(12n+7)q^n = 4\frac{f_3f_4^2f_6}{f_1^3}.$$
(44)

Using (17) we rewrite (44) as

$$\sum_{n=0}^{\infty} p_{\xi}(12n+7)q^n = 4\frac{f_4^8 f_6^4}{f_2^9 f_{12}^2} + 12q\frac{f_4^4 f_6^2 f_{12}^2}{f_2^7}.$$

Taking the odd parts on both sides of the last equation, we are left with

$$\sum_{n=0}^{\infty} p_{\xi} (24n+19)q^n = 12 \frac{f_2^4 f_3^2 f_6^2}{f_1^7},$$

which proves (41).

In order to prove (42), we use (22) to extract the terms of the form q^{3n} of (28). The resulting identity is

$$\sum_{n=0}^{\infty} p_{\xi}(9n)q^{3n} = \frac{f_6^2 f_9^6}{f_3^4 f_{18}^3},$$

which, after replacing q^3 by q and using (8), yields

$$\sum_{n=0}^{\infty} p_{\xi}(9n)q^n = \frac{f_2^2 f_3^3}{f_1^4 f_6^3} \equiv \frac{f_2^2 f_3^5}{f_1 f_6^3} = \psi(q) \frac{f_3^5}{f_6^3} \pmod{3}.$$

By (8), we have

$$\sum_{n=0}^{\infty} p_{\xi}(9n) q^n \equiv \frac{f_3^5}{f_6^3} \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod{3}.$$

Since $n(n+1)/2 \not\equiv 2 \pmod{3}$ for all $n \ge 0$, all terms of the form q^{3n+2} in the last expression have coefficients congruent to $0 \pmod{3}$, which proves (42).

We now prove (43). Replacing (22) in (28) and extracting the terms of the form q^{3n+2} , we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(9n+6)q^{3n+2} = 4q^2 \frac{f_{18}^3}{f_3^2}.$$
(45)

Dividing both sides of (45) by q^2 and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_{\xi}(9n+6)q^n = 4\frac{f_6^3}{f_1^2}.$$
(46)

Now we use (11) to extract the odd part of (46) and obtain

$$\sum_{n=0}^{\infty} p_{\xi} (18n+15)q^n = 8 \frac{f_2^2 f_3^3 f_8^2}{f_1^5 f_4}.$$

Since $f_1^3 \equiv f_3 \pmod{3}$, we have

$$\sum_{n=0}^{\infty} p_{\xi}(18n+15)q^n \equiv 2\frac{f_2^2 f_3^2 f_8^2}{f_1^2 f_4} \pmod{3}.$$

Using (15) we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(36n+15)q^n \equiv 2\frac{f_2^3 f_3 f_4 f_6^2}{f_1^3 f_{12}} \pmod{3}.$$

Since the odd part of (17) is divisible by 3, then the coefficients of the terms of the form q^{2n+1} in $\sum_{n=0}^{\infty} p_{\xi}(36n+15)q^n$ are congruent to 0 modulo 3. This completes the proof of (43).

We now prove a pair of unexpected congruences modulo 5 satisfied by $p_{\xi}(n)$.

Theorem 11 For all $n \ge 0$, we have

$$p_{\xi}(45n+33) \equiv 0 \pmod{5},\tag{47}$$

$$p_{\xi}(45n+42) \equiv 0 \pmod{5}.$$
 (48)

Proof By (46), we have

$$\sum_{n=0}^{\infty} p_{\xi}(9n+6)q^n = 4\frac{f_6^3}{f_1^2} = 4\frac{f_1^3f_6^3}{f_1^5} \equiv 4\frac{f_1^3f_6^3}{f_5} \pmod{5}.$$

Thanks to Jacobi's identity (10) we know

$$f_1^3 f_6^3 = \sum_{j,k=0}^{\infty} (-1)^{j+k} (2j+1)(2k+1)q^{3j(j+1)+k(k+1)/2}.$$

Note that, for all integers j and k, 3j(j+1) and k(k+1)/2 are congruent to either 0, 1 or 3 modulo 5. The only way to obtain 3j(j+1)+k(k+1)/2 = 5n+3is the following:

 $-3j(j+1) \equiv 0 \pmod{5}$ and $k(k+1)/2 \equiv 3 \pmod{5}$, or $-3j(j+1) \equiv 3 \pmod{5}$ and $k(k+1)/2 \equiv 0 \pmod{5}$.

Thus, $j \equiv 2 \pmod{5}$ or $k \equiv 2 \pmod{5}$ in all possible cases, and this means

$$(2j+1)(2k+1) \equiv 0 \pmod{5}.$$

Therefore, for all $n \ge 0$, $p_{\xi}(45n + 33) = p_{\xi}(9(5n + 3) + 6) \equiv 0 \pmod{5}$, which is (47).

In order to complete the proof of (48), we want to see when

$$3j(j+1) + k(k+1)/2 = 5n + 4.$$

Four possible cases arise:

- $-k \equiv 1 \pmod{5}$ and $j \equiv 2 \pmod{5}$, $-k \equiv 3 \pmod{5}$ and $j \equiv 2 \pmod{5}$,
- $-j \equiv 1 \pmod{5}$ and $k \equiv 2 \pmod{5}$,
- or $-j \equiv 3 \pmod{5}$ and $k \equiv 2 \pmod{5}$.

In all four cases above, either $j \equiv 2 \pmod{5}$ or $k \equiv 2 \pmod{5}$. So

$$(2j+1)(2k+1) \equiv 0 \pmod{5}$$

in all these cases. Therefore,

$$p_{\xi}(45n+42) = p_{\xi}(9(5n+4)+6) \equiv 0 \pmod{5},$$

which completes the proof of (48).

Next, we prove three congruences modulo 8 which are not covered by the above results.

Theorem 12 For all $n \ge 0$, we have

$$p_{\xi}(16n+14) \equiv 0 \pmod{8},\tag{49}$$

$$p_{\xi}(24n+13) \equiv 0 \pmod{8},$$
 (50)

$$p_{\xi}(24n+22) \equiv 0 \pmod{8}.$$
 (51)

Proof Initially we prove (49). From (34) and (7) we have

$$\sum_{n=0}^{\infty} p_{\xi}(4n+2)q^n \equiv \frac{f_3^2 f_5^5}{f_{12}^2} + \frac{f_{12}^2}{f_3^2 f_6} \phi(q)^2 \pmod{8}.$$

Now we can use (11), (12), and (20) to extract the terms involving q^{2n+1} from both sides of the previous congruence:

$$\sum_{n=0}^{\infty} p_{\xi}(8n+6)q^{2n+1} \equiv -2q^3 \frac{f_6^6 f_{48}^2}{f_{12}^2 f_{24}} + 2q^3 \frac{f_4^{10} f_{12}^4 f_{48}^2}{f_2^4 f_6^6 f_8^4 f_{24}} + 4q \frac{f_8^4 f_{12}^2 f_{24}^5}{f_4^2 f_6^6 f_{48}^2} \pmod{8}$$

After dividing both sides by q and then replacing q^2 by q, we are left with

$$\begin{split} \sum_{n=0}^{\infty} p_{\xi}(8n+6)q^n &\equiv -2q \frac{f_3^6 f_{24}^2}{f_6^2 f_{12}} + 2q \frac{f_2^{10} f_6^4 f_{24}^2}{f_1^4 f_3^6 f_4^4 f_{12}} + 4 \frac{f_4^4 f_6^2 f_{12}^5}{f_2^2 f_3^6 f_{24}^2} \\ &\equiv -2q \frac{f_3^6 f_{24}^2}{f_6^2 f_{12}} + 2q \frac{f_3^6 f_6^4 f_{24}^2}{f_3^{12} f_{12}} + 4 \frac{f_4^4 f_{12}^5}{f_2^2 f_6 f_{24}^2} \\ &\equiv 4 \frac{f_4^3 f_{12}}{f_6} \pmod{8}, \end{split}$$

whose odd part is congruent to 0 modulo 8, which implies (49).

In order to prove (50), we use (15) to obtain the even part of identity (40), which is

$$\sum_{n=0}^{\infty} p_{\xi} (12n+1)q^n = 2 \frac{f_2^3 f_6^4}{f_1^4 f_{12}^2}.$$

Now, employing (13), we obtain the odd part of the last identity, which is

$$\sum_{n=0}^{\infty} p_{\xi} (24n+13)q^n = 8 \frac{f_2^2 f_3^4 f_4^4}{f_1^7 f_6^2},$$

which implies (50).

Now we prove (51). We employ (15) in (39) to obtain

$$\sum_{n=0}^{\infty} p_{\xi} (12n+10)q^n = 4 \frac{f_2^3 f_3^2 f_{12}^2}{f_1^4 f_6^2}.$$
(52)

By (12) and (13), we rewrite (52) in the form

$$\sum_{n=0}^{\infty} p_{\xi}(12n+10)q^n = 4\frac{f_2^3 f_{12}^2}{f_6^2} \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) \left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right),$$

from which we obtain

$$\sum_{n=0}^{\infty} p_{\xi}(24n+22)q^{2n+1} = 4\frac{f_2^3 f_{12}^2}{f_6^2} \left(-2q^3 \frac{f_4^{14} f_6 f_{48}^2}{f_2^{14} f_8^4 f_{24}} + 4q \frac{f_4^2 f_6 f_8^4 f_{24}^5}{f_2^{10} f_{12}^2 f_{48}^2}\right).$$

Dividing both sides by q and replacing q^2 by q, we are left with

$$\sum_{n=0}^{\infty} p_{\xi}(24n+22)q^n = -8q \frac{f_2^{14}f_6^2f_{24}^2}{f_1^{11}f_3f_4^4f_{12}} + 16\frac{f_2^2f_4^4f_{12}^5}{f_1^7f_3f_{24}^2},$$

which implies (51).

We close this section by proving a congruence modulo 9.

Theorem 13 For all $n \ge 0$, we have

$$p_{\xi}(96n+76) \equiv 0 \pmod{9}.$$
 (53)

Proof We use (21) to extract the terms of the form q^{3n+1} from (32). The resulting identity is

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^{3n+1} = 2q \frac{f_6^6 f_9 f_{18}}{f_3^5 f_{12}^2},$$

which, after dividing by q and replacing q^3 by q, yields

$$\sum_{n=0}^{\infty} p_{\xi}(12n+4)q^n = 2\frac{f_2^6 f_3 f_6}{f_1^5 f_4^2} = 2\frac{f_2^6 f_6}{f_4^2} \frac{f_3}{f_1} \frac{1}{f_1^4}.$$

Using (13) and (14), we extract the even part on both sides of the above identity to obtain

$$\sum_{n=0}^{\infty} p_{\xi}(24n+4)q^n = 2\frac{f_{2}^{13}f_3^2f_8f_{12}^2}{f_1^{10}f_4^5f_6f_{24}} + 8q\frac{f_3^2f_4^6f_{24}}{f_1^6f_8f_{12}}$$
$$\equiv 2\frac{f_{2}^{13}f_8f_{12}^2}{f_4^5f_6f_{24}}\frac{1}{f_1f_3} + 8q\frac{f_4^6f_{24}}{f_8f_{12}}\frac{f_1^3}{f_3} \pmod{9}.$$

Now we employ (18) and (16) to extract the odd part on both sides of the last congruence:

$$\sum_{n=0}^{\infty} p_{\xi}(48n+28)q^n \equiv 2\frac{f_1^9 f_6 f_{12}}{f_3^3 f_4} + 8\frac{f_2^9 f_{12}}{f_4 f_6^2} \equiv \frac{f_6 f_{12}}{f_4} \pmod{9},$$

which implies (53).

6 Concluding remarks

Computational evidence indicates that $p_{\xi}(n)$ satisfies many other congruences. The interested reader may wish to consider the following two conjectures.

Conjecture 1

$$\sum_{n=0}^{\infty} p_{\xi}(8n+3)q^n \equiv 2\sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod{3}$$

Conjecture 2

$$\sum_{n=0}^{\infty} p_{\xi}(32n+12)q^n \equiv 6\sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod{9}$$

Clearly, once proven, Conjectures 1 and 2 would immediately lead to infinite families of Ramanujan–like congruences. Morever, Conjecture 2 would immediately imply Theorem 13 since 96n + 76 = 32(3n + 2) + 12 while the right–hand side of Conjecture 2 is clearly a function of q^3 . The same argument would imply that, for all $n \ge 0$,

$$p_{\xi}(96n+44) \equiv 0 \pmod{9}$$

since 96n + 44 = 32(3n + 1) + 12.

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