Relative Equilibria in the Four-Vortex Problem with Two Pairs of Equal Vorticities

**Marshall Hampton · Gareth E. Roberts ·
Manuele Santoprete**

Abstract We examine in detail the relative equilibria in the planar four-vortex problem where two pairs of vortices have equal strength, that is, $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$ where $m \in \mathbb{R} - \{0\}$ is a parameter. One main result is that, for $m > 0$, the convex configurations all contain a line of symmetry, forming a rhombus or an isosceles trapezoid. The rhombus solutions exist for all m but the isosceles trapezoid case exists only when m is positive. In fact, there exist asymmetric convex configurations when $m < 0$. In contrast to the Newtonian four-body problem with two equal pairs of masses, where the symmetry of all convex central configurations is unproven, the equations in the vortex case are easier to handle, allowing for a complete classification of all solutions. Precise counts on the number and type of solutions (equivalence classes) for different values of m, as well as a description of some of the bifurcations that occur, are provided. Our techniques involve a combination of analysis, and modern and computational algebraic geometry.

M. Hampton
Department of Mathematics and Statistics, University of Minnesota Duluth, 1117 University Drive,
Duluth, MN 55812-3000, USA

G.E. Roberts (✉)
Department of Mathematics and Computer Science, College of the Holy Cross, 1 College Street,
Worcester, MA 01610, USA
e-mail: groberts@holycross.edu

M. Santoprete
Department of Mathematics, Wilfrid Laurier University, 75 University Avenue West, Waterloo,
Ontario, Canada, N2L 3C5

1 Introduction

The motion of point vortices in the plane is an old problem in fluid mechanics that
was first given a Hamiltonian formulation by Kirchhoff (1883). This widely used
model provides finite-dimensional approximations to vorticity evolution in fluid dy-
namics. The goal is to track the motion of the point vortices rather than focus on their
internal structure and deformation, a concept analogous to the use of "point masses"
in celestial mechanics. As with the Newtonian n-body problem, an important class
of homographic solutions exist where the configuration is self-similar throughout the
motion. Such solutions are described as *stationary* by O'Neil (1987), or called *vortex
crystals* by Aref et al. (2003), and are not limited to just relative equilibria, but also
include equilibria, rigidly translating solutions, and collapse configurations.

A relative equilibrium is a special configuration of vortices that rigidly rotates
about its center of vorticity. In a rotating coordinate system it becomes a rest point,
hence the name. The first and perhaps most well-known example occurs when three
vortices are placed at the vertices of an equilateral triangle. Originally discovered by
Lord Kelvin in 1867 (Lord Kelvin 1867) for identical vortices, the equilateral trian-
gle solution is a relative equilibrium for any choice of vorticities. Many examples,
particularly symmetric configurations, have been discovered and analyzed from a va-
riety of perspectives (see Aref et al. 2003 or Newton 2001 and the references therein
for a good overview on the subject). For example, Aref (2007b) used special "gener-
ating polynomials," where the vortices are located at the roots of the polynomial, to
study symmetric examples of equal-strength vortices such as nested regular polygons.
Newton and Chamoun (2007) used the singular value decomposition and a Brownian
"ratchet" algorithm to numerically locate several examples of asymmetric stationary
solutions. Recently, Chen et al. (2013) utilized an aggregation model and a mean-field
limit to find stable relative equilibria in the planar n-vortex problem for large values
of n. Other notable approaches include topological methods developed by Palmore
(1982), and later O'Neil (1987), to count or bound the number of solutions, and the
application of BKK theory by Moeckel and Hampton (2009) to prove finiteness and
obtain upper bounds for the number of relative equilibria in the four-vortex problem.

In this paper we use techniques from modern and computational algebraic ge-
ometry to study the relative equilibria of the four-vortex problem when two pairs of
vortices have the same vortex strength. Specifically, if $\Gamma_i \in \mathbb{R} - \{0\}$ denotes the vortex
strength of the ith vortex, we set $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$, treating m as a real
parameter. Our main goal is to classify and describe all solutions as m varies. Pla-
nar four-vortex configurations that are not collinear are called *strictly planar*. Strictly
planar configurations that do not contain any three vortices which are collinear can
be classified as either *concave* or *convex*. A concave configuration has one vortex
which is located strictly inside the convex hull of the other three, whereas a convex
configuration does not have a vortex contained in the convex hull of the other three

vortices. A configuration is called a *kite* if two vortices are on an axis of symmetry and the other two vortices are symmetrically located with respect to this axis. Kite configurations may either be concave or convex.

The symmetry and asymmetry of solutions also plays a major role in our analysis. In fact, part of the motivation behind our work was to determine whether symmetry could be proven for this choice of vorticities when $m > 0$ and the configuration is assumed to be convex. This question, while solved in the Newtonian four-body problem when the equal masses are assumed to be opposite each other in a convex central configuration (Albouy et al. 2008; Perez-Chavela and Santoprete 2007), is still open for the case when equal masses are assumed to be adjacent.
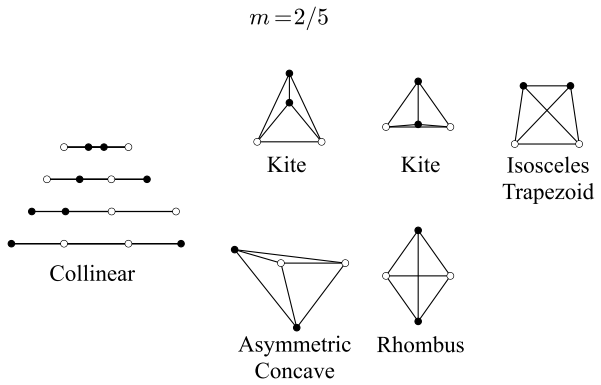
1.1 Summary of Main Results

One of the main results in this work is proving that any convex relative equilibrium with $m > 0$, and any concave solution with $m < 0$, must have a line of symmetry. This, in turn, allows us to describe the solutions in great detail. For the convex case and $m > 0$, the symmetric solutions are a rhombus and an isosceles trapezoid. In the concave case and $m < 0$, there is a symmetric solution only when $-1/2 < m < 0$. This consists of an isosceles triangle with an interior vortex on the axis of symmetry.

Another major result of our work is a precise count on the number and type of solutions for different values of m (see Table 1). Without loss of generality, we restrict to the case where $m \in (-1, 1]$. The choice $m = -1$ implies that $\Gamma = \sum_i \Gamma_i = 0$, a
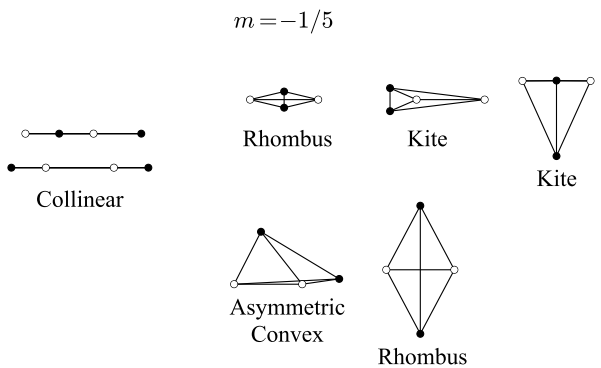
Table 1 The third column of the table lists the number of relative equilibria equivalence classes for the four-vortex problem with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$, in terms of m and the type of configuration. The last column lists the corresponding theorem or lemma in this paper. The special value $m^* \approx -0.5951$ is the only real root of the cubic $9m^3 + 3m^2 + 7m + 5$. Kite$_{ij}$ refers to a kite configuration with vortices i and j on the axis of symmetry

Shape	$m \in (-1, 1]$	Type of solution (number of)	Theorem(s)
Convex	$m = 1$	Square (6)	7.1, 7.7
	$0 < m < 1$	Rhombus (2), Isosceles Trapezoid (4)	7.1, 7.7
	$-1 < m < 0$	Rhombus (4)	7.7
		Asymmetric (8)	6.10
	$-1/2 < m < 0$	Kite$_{34}$ (4)	7.2
	$m^* < m < -1/2$	Kite$_{12}$ (4)	7.6
Concave	$m = 1$	Equilateral Triangle with Interior Vortex (8)	7.2
	$0 < m < 1$	Kite$_{34}$ (8)	7.2
		Asymmetric (8)	6.10
	$-1/2 < m < 0$	Kite$_{12}$ (4)	7.2
Collinear	$m = 1$	Symmetric (12)	5.2
	$0 < m < 1$	Symmetric (4)	5.2
		Asymmetric (8)	5.2
	$-1 < m < 0$	Symmetric (2)	5.2
	$-1/2 < m < 0$	Asymmetric (4)	5.2

Fig. 1 The geometrically
distinct solutions for $m = \frac{2}{5}$

$m = 2/5$

Kite   Kite   Isosceles
                  Trapezoid

Collinear

Asymmetric   Rhombus
Concave

Fig. 2 The geometrically
distinct solutions for $m = -\frac{1}{5}$

$m = -1/5$

Rhombus   Kite

                              Kite

Collinear

Asymmetric
Convex

Rhombus

special case examined in Sect. 4.2. For any $m < -1$ or $m > 1$, we can rescale the circulations and relabel the vortices to return to a particular case with $m \in (-1, 1]$.
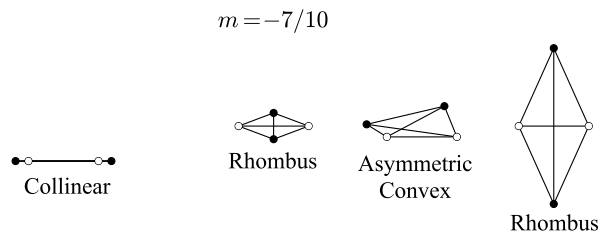
When counting solutions, we treat each configuration as a distinct point in $\mathbb{R}^8$ and use the standard convention from celestial mechanics that solutions which are identical under scaling or rotation are considered equivalent. However, two solutions identical under a reflection are counted separately. Due to the symmetry in the choice of circulations, we may generate new solutions by simply interchanging vortices 1 and 2, or vortices 3 and 4, or both. Thus, in general, any particular configuration generates four solutions. Additionally, if it does not have a line of symmetry, then we can reflect each of these solutions to arrive at a count of eight. For symmetric configurations, we must reduce the count depending on how much symmetry is present. For example, when $m > 0$, there is a unique isosceles trapezoid configuration that leads to a count of four solutions once all possible permutations are considered (reflecting about the line of symmetry is equivalent to interchanging both pairs of vortices). On the other hand, there is a unique rhombus solution for $m > 0$, but this only gives a count of two due to equivalence after a 180° rotation.

The geometrically distinct solutions (not identical under a relabeling, rotation or reflection) for $m = 2/5$, $m = -1/5$, $m = -0.55$, and $m = -7/10$ are shown in Figs. 1, 2, 3 and 4, respectively. Vortices 1 and 2 are denoted by open disks ($\Gamma_1 = \Gamma_2 = 1$), while vortices 3 and 4 are identified by solid disks ($\Gamma_3 = \Gamma_4 = m$).

Fig. 3 The geometrically distinct solutions for $m = -0.55$

$m = -11/20$

Collinear

Rhombus

Kite

Asymmetric Convex

Rhombus
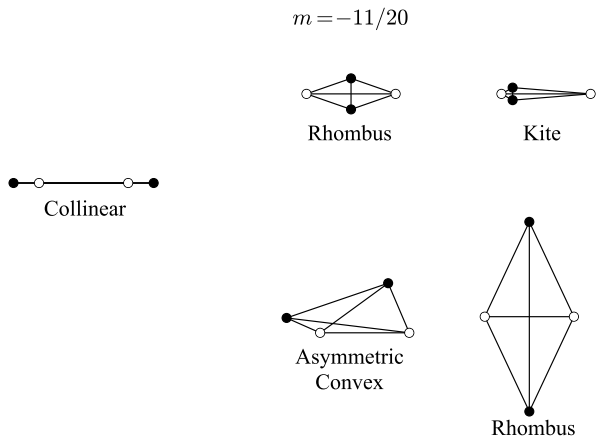
Fig. 4 The geometrically distinct solutions for $m = -\frac{7}{10}$

$m = -7/10$

Collinear

Rhombus

Asymmetric Convex

Rhombus

When $m = 1$ (all vortex strengths equal), there are 26 solutions, all symmetric. Allowing for relabeling, there are only three geometrically distinct configurations: a square, an equilateral triangle with a vortex at the center, and a collinear configuration. This is different from the Newtonian case, where an additional symmetric, concave solution exists, consisting of an isosceles triangle with an interior body on the axis of symmetry (Albouy 1996). The reason for the contrast is that the equilateral triangle with central vortex is degenerate when all vortices have the same strength (Albouy 1996; Meyer and Schmidt 1988; Palmore 1982). An interesting bifurcation occurs as m decreases through $m = 1$, where the equilateral triangle solution splits into four different solutions. If vortex 3 or 4 is at the center of the triangle, then the solution for $m = 1$ bifurcates into two different isosceles triangles with the interior vortex on the line of symmetry. In this case, the larger-strength vortices are on the base of the triangle. If vortex 1 or 2 is at the center of the triangle, the solution branches into two asymmetric concave configurations that are identical under a reflection. Thus, the number of solutions increases from 26 to 34 for the case $0 < m < 1$. This concurs with Palmore's result in Palmore (1982), which specifies a lower bound of 34 *non-degenerate* solutions for the four-vortex problem with positive vorticities.

An interesting geometric feature of our problem is the continuous transformation from concave to convex configurations as m becomes negative. One of the concave kite families with vortices 3 and 4 on the line of symmetry becomes a convex kite as m decreases through 0. At $m = 0$, three vortices become collinear (either vortex 3 or vortex 4 lies between vortices 1 and 2). The same phenomenon occurs for the asymmetric configurations: as m decreases through 0, the asymmetric solutions

continuously change from concave to convex, passing through a "boundary" configuration at $m = 0$ where three vortices are collinear. Not all solutions persist as m becomes negative. For example, the isosceles trapezoid family disappears as m becomes negative (solutions become complex). Moreover, some solutions, such as the rhombus, keep their shape even after m flips sign.

For $m < 0$ there are two interesting bifurcation values at $m = -1/2$ and $m = m^* \approx -0.5951$, the only real root of the cubic $9m^3 + 3m^2 + 7m + 5$. There are 26 solutions for $-1/2 < m < 0$, including a family of concave kite configurations having one of the positive strength vortices in the interior of an isosceles triangle formed by the outer three. As m approaches $-1/2$, the triangle formed by the interior vortex and the base of the outer isosceles triangle limits on an equilateral triangle while the fourth vortex (at the apex of the outer isosceles triangle) heads off to infinity. For $m < -1/2$, this family bifurcates into a convex configuration. This new family of convex kites, with vertices 1 and 2 on the axis of symmetry, exists only for $m^* < m < -1/2$. In sum, there are 18 solutions for $m^* < m < -1/2$, and 14 solutions when $m = -1/2$ or when $-1 < m \leq m^*$. The singular bifurcation at $m = -1/2$ occurs in part because the leading coefficient of a defining polynomial vanishes. This is likely a consequence of the fact that the sum of three vorticities vanishes when $m = -1/2$, a particularly troubling case when attempting to prove finiteness for the number of relative equilibria in the full four-vortex problem (see Hampton and Moeckel 2009). The bifurcation at $m = m^*$ is a pitchfork bifurcation, as discussed in Sect. 7.4.

1.2 Comparisons with the Newtonian n-Body Problem

There are some key differences between the vortex case and the n-body problem. First, in contrast with the Newtonian case, it is possible to have actual equilibria (rest points of the flow) in the n-vortex problem. This can only happen when the total vortex angular momentum $\sum_{i<j} \Gamma_i \Gamma_j$ vanishes, a condition that requires circulations of opposite signs. For $n = 4$, explicit solutions can be derived (see Hampton and Moeckel 2009). An analysis of the equilibrium solutions in the context of symmetry is presented in Sect. 4.1.

Another difference occurs for the special case $m = 1$. As described above, while both problems have the square and a symmetric collinear configuration as solutions, the only concave relative equilibrium for $m = 1$ is the equilateral triangle with a vortex at the center and this solution is degenerate (not isolated after eliminating the rotational symmetry). This contrasts with the equal mass four-body problem where an additional concave kite exists (Albouy 1996) and the equilateral triangle solution is non-degenerate.

Some expected differences arise due to the fact that the vorticities can be of opposite sign. For example, there are no concave relative equilibria when $m \in (-1, -1/2)$. This is opposite the result of Hampton (2002) that states for any choice of four positive masses, there exists a concave central configuration. Another contrast comes from the fact that there are two geometrically distinct rhombi when $m < 0$ (discussed in Sect. 7.4) as well as a convex kite for $m^* < m < -1/2$. This means that there is not a unique (up to symmetry) convex relative equilibrium when $m < 0$. This contrasts with the Newtonian four-body problem where it is thought (although unproven) that

there is a unique convex central configuration for any fixed ordering of four positive masses (Problem 10 in Albouy et al. 2012).

Here is the outline for the remainder of the paper. In the next section we define a relative equilibrium and explain how to use mutual distances as variables in the planar four-vortex problem. In Sect. 3, we describe the relevant algebraic techniques used to analyze and quantify the number of solutions. Section 4 examines the interplay between symmetry and equality of vorticities in two special cases: equilibria and vanishing total vorticity. Section 5 examines the collinear case in detail. Our main symmetry theorem is proved in Sect. 6 along with a proof of the existence of asymmetric solutions. The different symmetric cases are covered in Sect. 7. Throughout our research, symbolic computations (e.g., calculation of Gröbner bases) were performed using Sage (Stein et al. 2011), SINGULAR (Decker et al. 2011) and Maple15^{TM}.

2 Relative Equilibria

We begin with the equations of motion and the definition of a relative equilibrium for the planar n-vortex problem. Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and let ∇_j denote the two-dimensional partial gradient with respect to x_j. A system of n point vortices with vortex strengths $\Gamma_i \neq 0$ and positions $x_i \in \mathbb{R}^2$ evolves according to

$$\Gamma_i \dot{x}_i = J \nabla_i H = -J \sum_{j=1}^{n} {}' \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(x_i - x_j), \quad 1 \leq i \leq n, \tag{1}$$

where $H = -\sum_{i<j} \Gamma_i \Gamma_j \log r_{ij}$, $r_{ij} = \|x_i - x_j\|$, and the prime on the summation indicates omission of the term with $j = i$.

A *relative equilibrium* motion is a solution of (1) of the form $x_i(t) = c + e^{-J\lambda t}(x_i(0) - c)$, that is, a periodic solution given by a uniform rotation with angular velocity $\lambda \neq 0$ around some point $c \in \mathbb{R}^2$. Such a solution is possible if and only if the initial positions $x_i(0)$ satisfy the equations

$$-\lambda(x_i(0) - c) = \frac{1}{\Gamma_i} \nabla_i H = \sum_{j=1}^{n} {}' \frac{\Gamma_j}{r_{ij}^2}(x_j(0) - x_i(0)) \tag{2}$$

for each $i \in \{1, \ldots, n\}$. Denote $\Gamma = \sum_i \Gamma_i$ as the *total circulation* and assume for the moment that $\Gamma \neq 0$. Multiplying the ith equation in (2) by Γ_i and summing over i shows that the center of rotation c is equivalent to the *center of vorticity*, $c = \frac{1}{\Gamma} \sum_i \Gamma_i x_i$. If $\Gamma = 0$, then we obtain instead that the *moment of vorticity* $\sum_i \Gamma_i x_i$ must vanish.

Definition 2.1 A set of initial positions $x_i(0)$ satisfying Eq. (2) for each $i \in \{1, \ldots, n\}$ is called a *central configuration*. The corresponding rigid rotation with angular velocity $\lambda \neq 0$ is called a *relative equilibrium*. We will often use these two terms interchangeably.

Note that if $(x_1, x_2, \ldots, x_n)$ is a central configuration, then so is $(\kappa x_1, \kappa x_2, \ldots, \kappa x_n)$ for any scalar κ. Here, the angular velocity λ must be scaled by a factor of $1/\kappa^2$. It is also readily apparent from Eq. (2) that $(e^{J\theta} x_1, e^{J\theta} x_2, \ldots, e^{J\theta} x_n)$ is also a central configuration for any angle θ with the same angular velocity λ. Thus, relative equilibria are not isolated solutions. As described in the introduction, we handle these symmetries by fixing the scaling and identifying configurations that are equivalent under rotation. In addition, it is possible to reflect each vortex about one of the coordinate axes since, for example, multiplying the first component of each x_i by -1 still generates a solution to Eq. (2). We count reflected solutions separately unless they are equivalent under a rotation.

Define the *angular impulse I* with respect to the center of vorticity as

$$I = \frac{1}{2} \sum_{i=1}^{n} \Gamma_i \|x_i - c\|^2.$$

I measures the size of the system and is the analog of the moment of inertia in the n-body problem. It is well known that I is a conserved quantity in the planar n-vortex problem (Newton 2001). We can then rewrite Eq. (2) as

$$\nabla H + \lambda \nabla I = 0, \tag{3}$$

where $\nabla = (\nabla_1, \ldots, \nabla_n)$. Therefore, λ can be viewed as a Lagrange multiplier and any solution of (2) can be interpreted as a critical point of the Hamiltonian $H(x)$ under the condition that I remains constant. Using the homogeneity of the functions H and I, Eq. (3) implies that the angular velocity λ in a relative equilibrium is given by

$$\lambda = \frac{L}{2I}, \quad \text{where } L = \sum_{i<j}^{n} \Gamma_i \Gamma_j \tag{4}$$

is the *total vortex angular momentum*. This implies that in the case where all vortex strengths are positive, $\lambda > 0$ and the relative equilibrium is rotating in the counterclockwise direction. It is also important to note that, for any relative equilibrium with a particular set of vortex strengths Γ_i, we can scale the vorticities by some common factor ν and maintain the relative equilibrium, but with a new angular velocity $\nu\lambda$. If $\nu < 0$, then the sign of λ flips, as does the direction of rotation.

2.1 Using Mutual Distances as Coordinates

We now consider the case of $n = 4$ vortices. Our presentation follows the approach of Schmidt (2002) in describing the work of Dziobek (1900) for the Newtonian n-body problem. We want to express Eq. (3) in terms of the mutual distance variables r_{ij}. Between four vortices there are six mutual distances, which, in general, define a tetrahedron. If the configuration is to lie in a plane, then the volume of this tetrahedron must be zero. This condition is usually expressed in terms of the Cayley–Menger determinant. The Cayley–Menger determinant for the volume V is an expression that

gives the volume of a tetrahedron in terms of the lengths of its edges:

$$e_{CM} = 288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

Therefore, planar central configurations can be viewed as critical points of H under the condition that the moment of inertia remains constant and the Cayley–Menger determinant vanishes, i.e., $e_{CM} = 0$. Let μ and λ be Lagrange multipliers. Then, planar central configurations are obtained as critical points of

$$H + \lambda(I - I_0) + \frac{\mu}{32} e_{CM} \tag{5}$$

with the constraints $I = I_0$ and $e_{CM} = 0$, where I_0 is some fixed constant value of the moment of inertia. Using the homogeneity of H, I, and e_{CM}, the value of λ in this setup is identical to one given in Eq. (4).

To find ∇e_{CM} restricted to planar configurations, we use the following important formula:

$$\frac{\partial e_{CM}}{\partial r_{ij}^2} = -32 A_i A_j,$$

where A_i is the oriented area of the triangle T_i whose vertices are all the vortices except for the ith vortex. Here, the signs of the oriented areas can be obtained by using the following convention. First, list the points of the triangle in counterclockwise order and append the missing point to the end of the list to form a permutation $\phi \in S_4$. Then, the sign of ϕ, that is, the sign of the permutation that takes (1234) to ϕ, is the sign of A_i. Note that from our point of view, the opposite convention (clockwise order) is equivalent, since our equations only depend on the product of the oriented areas. One important property of this sign convention is that $A_1 + A_2 + A_3 + A_4 = 0$.

To give an idea of how the sign convention works, suppose that the four vortices form a concave configuration with the fourth vortex in the convex hull of the other three. Then we either have $A_4 > 0$ and $A_1, A_2, A_3 < 0$, or $A_4 < 0$ and $A_1, A_2, A_3 > 0$. On the other hand, if we have a convex configuration where the first and second vortices are opposite each other, then A_1 and A_2 will have the same sign, and A_3 and A_4 will have the same sign, but the signs of each pair will be opposite. In general, for a concave configuration with the ith vortex in the interior, A_i has one sign and the remaining three oriented areas have the opposite sign. For a convex configuration with vortices i and j opposite each other, A_i and A_j have the same sign, and the remaining two oriented areas have the opposite sign. This concurs with the set up described in Schmidt (2002).

Setting the gradient of (5) equal to zero yields the equations

$$\frac{\partial H}{\partial r_{ij}^2} + \lambda \frac{\partial I}{\partial r_{ij}^2} + \frac{\mu}{32} \frac{\partial e_{CM}}{\partial r_{ij}^2} = 0.$$

If $\Gamma \neq 0$, then I can be written in terms of the mutual distances as

$$I = \frac{1}{2\Gamma} \sum_{i<j} \Gamma_i \Gamma_j r_{ij}^2.$$

Using this, we obtain the following equations for a four-vortex central configuration:

$$\Gamma_i \Gamma_j \left(r_{ij}^{-2} + \lambda' \right) = \sigma A_i A_j, \tag{6}$$

where $\lambda' = -\lambda/\Gamma$, $\sigma = -2\mu$, $I = I_0$, and $e_{CM} = 0$. If $\Gamma = 0$, then a different approach is more useful (see Celli 2005; Hampton and Moeckel 2009 for expositions of the equations). We discuss the role of specific symmetries when $\Gamma = 0$ in Sect. 4.2.

Assuming $\Gamma \neq 0$, we group the equations in (6) so that when they are multiplied together pairwise, their right-hand sides are identical:

$$\Gamma_1 \Gamma_2 \left(r_{12}^{-2} + \lambda' \right) = \sigma A_1 A_2, \qquad \Gamma_3 \Gamma_4 \left(r_{34}^{-2} + \lambda' \right) = \sigma A_3 A_4,$$

$$\Gamma_1 \Gamma_3 \left(r_{13}^{-2} + \lambda' \right) = \sigma A_1 A_3, \qquad \Gamma_2 \Gamma_4 \left(r_{24}^{-2} + \lambda' \right) = \sigma A_2 A_4, \tag{7}$$

$$\Gamma_1 \Gamma_4 \left(r_{14}^{-2} + \lambda' \right) = \sigma A_1 A_4, \qquad \Gamma_2 \Gamma_3 \left(r_{23}^{-2} + \lambda' \right) = \sigma A_2 A_3.$$

This yields the well-known Dziobek equations (Dziobek 1900), but for vortices

$$\left(r_{12}^{-2} + \lambda' \right)\left(r_{34}^{-2} + \lambda' \right) = \left(r_{13}^{-2} + \lambda' \right)\left(r_{24}^{-2} + \lambda' \right) = \left(r_{14}^{-2} + \lambda' \right)\left(r_{23}^{-2} + \lambda' \right). \tag{8}$$

From the different ratios of two vorticities that can be derived from (7), we obtain the following set of equations:

$$
\begin{aligned}
\frac{\Gamma_1 A_2}{\Gamma_2 A_1} &= \frac{\rho_{23} + \lambda'}{\rho_{13} + \lambda'} = \frac{\rho_{24} + \lambda'}{\rho_{14} + \lambda'} = \frac{\rho_{23} - \rho_{24}}{\rho_{13} - \rho_{14}}, \\
\frac{\Gamma_1 A_3}{\Gamma_3 A_1} &= \frac{\rho_{23} + \lambda'}{\rho_{12} + \lambda'} = \frac{\rho_{34} + \lambda'}{\rho_{14} + \lambda'} = \frac{\rho_{23} - \rho_{34}}{\rho_{12} - \rho_{14}}, \\
\frac{\Gamma_1 A_4}{\Gamma_4 A_1} &= \frac{\rho_{24} + \lambda'}{\rho_{12} + \lambda'} = \frac{\rho_{34} + \lambda'}{\rho_{13} + \lambda'} = \frac{\rho_{24} - \rho_{34}}{\rho_{12} - \rho_{13}}, \\
\frac{\Gamma_2 A_3}{\Gamma_3 A_2} &= \frac{\rho_{13} + \lambda'}{\rho_{12} + \lambda'} = \frac{\rho_{34} + \lambda'}{\rho_{24} + \lambda'} = \frac{\rho_{13} - \rho_{34}}{\rho_{12} - \rho_{24}}, \\
\frac{\Gamma_2 A_4}{\Gamma_4 A_2} &= \frac{\rho_{14} + \lambda'}{\rho_{12} + \lambda'} = \frac{\rho_{34} + \lambda'}{\rho_{23} + \lambda'} = \frac{\rho_{14} - \rho_{34}}{\rho_{12} - \rho_{23}}, \\
\frac{\Gamma_3 A_4}{\Gamma_4 A_3} &= \frac{\rho_{14} + \lambda'}{\rho_{13} + \lambda'} = \frac{\rho_{24} + \lambda'}{\rho_{23} + \lambda'} = \frac{\rho_{14} - \rho_{24}}{\rho_{13} - \rho_{23}},
\end{aligned} \tag{9}
$$

where $\rho_{ij} = r_{ij}^{-2}$.

Eliminating λ' from Eq. (8) and factoring yields the important relation

$$\left(r_{13}^2 - r_{12}^2 \right)\left(r_{23}^2 - r_{34}^2 \right)\left(r_{24}^2 - r_{14}^2 \right) = \left(r_{12}^2 - r_{14}^2 \right)\left(r_{24}^2 - r_{34}^2 \right)\left(r_{13}^2 - r_{23}^2 \right). \tag{10}$$

Assuming the six mutual distances determine an actual configuration in the plane, this equation is necessary and sufficient for the existence of a four-vortex relative equilibrium. The ratios of the corresponding vortex strengths are then found from the equations in (9).

The sign of $\lambda' = -\lambda/\Gamma$ for a given relative equilibrium is of some interest. Note that scaling the vorticities by any $\nu \in \mathbb{R} - \{0\}$ does *not* change the value of λ'. If all the vorticities have the same sign, then $\lambda' < 0$ is ensured. However, if the vorticities have different signs, it is possible that λ' could be positive.

When $\lambda' > 0$, the equations in (9) imply that $\frac{\Gamma_i A_j}{\Gamma_j A_i} > 0$ for any choice of indices i and j. Taking $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$ with $m < 0$, we see that the only possible solutions have $A_1, A_2 > 0$ and $A_3, A_4 < 0$, or $A_1, A_2 < 0$ and $A_3, A_4 > 0$. The configuration must be convex with vortices 1 and 2 on one diagonal and vortices 3 and 4 on the other. We show in Sect. 7.3 that the configuration must have at least one axis of symmetry. There exists a family of convex kite configurations for $m^* < m < -1/2$ and a family of rhombi (see Sect. 7.4) for $-1 \le m < -2 + \sqrt{3}$. These are the only possible solutions to our problem having $\lambda' > 0$.

If $\Gamma_i \ne 0$ for each i, then a strictly planar central configuration is either concave or convex. To see this, suppose by contradiction that vortices 1, 2 and 3 are collinear, but vortex 4 does not lie on their common line. Then, $A_4 = 0$ and (7) imply that $\Gamma_i \Gamma_4 = 0$ for at least one choice of $i \in \{1, 2, 3\}$, otherwise the configuration could not be realized geometrically. The argument is the same regardless of which three vortices are assumed to be collinear.

Proposition 2.2 *Suppose we have a concave central configuration with four vortices and $\Gamma \ne 0$.*

1. *If all the vorticities are positive (negative), then all of the exterior sides are longer than all of the interior ones, and $1/\sqrt{-\lambda'}$ is less than the lengths of the exterior sides and greater than the lengths of the interior sides.*
2. *If two of the vorticities are positive and two are negative, then the exterior sides connecting vortices with vorticities of opposite sign and the interior side connecting vortices with vorticities of the same sign have lengths greater than $1/\sqrt{-\lambda'}$, while the remaining sides have lengths less than $1/\sqrt{-\lambda'}$.*

Proof Note that $\lambda' < 0$ since the central configuration is assumed to be concave.

(1) Let Γ_4 be the interior vorticity and suppose that $A_4 < 0$. Then we have $A_1, A_2, A_3 > 0$. Assume that $\rho_{34} > -\lambda'$. Then, using the equations in (9), we obtain

$$\rho_{12}, \rho_{13}, \rho_{23} < -\lambda' < \rho_{14}, \rho_{24}, \rho_{34},$$

or

$$r_{12}, r_{13}, r_{23} > 1/\sqrt{-\lambda'} > r_{14}, r_{24}, r_{34},$$

namely all the exterior edges are longer than the interior ones. On the other hand, if we had assumed $\rho_{34} < -\lambda'$, then the inequalities above would be reversed and the configuration could not be realized geometrically since the interior sides cannot all be longer than the exterior sides.

(2) Suppose that $\Gamma_1, \Gamma_2 > 0$ and $\Gamma_3, \Gamma_4 < 0$. First, let Γ_4 be the interior vorticity and suppose that $A_4 < 0$. Then we have $A_1, A_2, A_3 > 0$. If $\rho_{34} < -\lambda'$, then, using the equations in (9), we obtain

$$\rho_{13}, \rho_{23}, \rho_{34} < -\lambda' < \rho_{12}, \rho_{14}, \rho_{24},$$

or

$$r_{13}, r_{23}, r_{34} > 1/\sqrt{-\lambda'} > r_{12}, r_{14}, r_{24}.$$

If we had assumed $\rho_{34} > -\lambda'$, then the inequalities above would be reversed and the configuration could not be realized geometrically. We show this with a proof by contradiction. Assume $\rho_{34} > -\lambda'$. Then $|A_4| < |A_3|$ since the two triangles have one side in common and $r_{24}, r_{14} > r_{23}, r_{13}$. But this is absurd since T_4 contains T_3.

Next, suppose that Γ_1 is the interior vorticity and that $A_1 < 0$. Then we have $A_2, A_3, A_4 > 0$. If $\rho_{34} > -\lambda'$ then, using the equations in (9), we obtain

$$\rho_{12}, \rho_{23}, \rho_{24} < -\lambda' < \rho_{13}, \rho_{14}, \rho_{34},$$

or

$$r_{12}, r_{23}, r_{24} > 1/\sqrt{-\lambda'} > r_{13}, r_{14}, r_{34}.$$

If we had assumed $\rho_{34} < -\lambda'$, then the inequalities above would be reversed, and the configurations could not be realized geometrically. (The proof is similar to the argument in the previous paragraph.) □

Before we state an analogous theorem for the convex case, we recall the following useful geometric lemma:

Lemma 2.3 *The combined length of the diagonals of a convex quadrilateral is greater than the combined length of any pair of its opposite sides.*

Proof Order the vortices counterclockwise. Let o be the position vector describing the intersection of the diagonals, and let $r_{io} = \|x_i - o\|$. Applying the triangle inequality to the triangle with vertices x_1, x_2, and o, and to the triangle with vertices x_3, x_4, and o, yields

$$r_{1o} + r_{2o} > r_{12} \quad \text{and} \quad r_{3o} + r_{4o} > r_{34}.$$

Adding these two inequalities together, we obtain $r_{13} + r_{24} > r_{12} + r_{34}$. A similar reasoning can be applied to the remaining two triangles. □

Proposition 2.4 *Suppose we have a convex central configuration with four vortices, with $\Gamma \neq 0$ and $\lambda' < 0$.*

1. *If all the vorticities are positive (negative), then all exterior sides are shorter than the diagonals. Furthermore, the lengths of all exterior sides are less than $1/\sqrt{-\lambda'}$ and the lengths of all the diagonals are greater than $1/\sqrt{-\lambda'}$. The shortest and longest exterior sides have to face each other.*

2. *If two of the vorticities are positive and two are negative, and the positive strength vortices are adjacent, then the exterior sides connecting vortices with vorticities of opposite sign have length less than $1/\sqrt{-\lambda'}$. All the other sides have length greater than $1/\sqrt{-\lambda'}$.*
3. *If two of the vorticities are positive and two are negative, and the positive strength vortices are opposite each other, then either all of the sides have length less than $1/\sqrt{-\lambda'}$, or all of the sides have length greater than $1/\sqrt{-\lambda'}$.*

Proof (1) The proof is analogous to the one for the Newtonian four-body problem (see Schmidt 2002).

(2) Order vortices $1, 2, 3, 4$ counterclockwise and let $\Gamma_1, \Gamma_2 > 0$ and $\Gamma_3, \Gamma_4 < 0$. The sign of the four areas are then $A_1 < 0$, $A_3 < 0$ and $A_2 > 0$, $A_4 > 0$. If we assume that $\rho_{34} > -\lambda'$, then we find from the equations in (9) that

$$\rho_{12}, \rho_{13}, \rho_{24}, \rho_{34} < -\lambda' < \rho_{14}, \rho_{23},$$

or

$$r_{12}, r_{13}, r_{24}, r_{34} > 1/\sqrt{-\lambda'} > r_{14}, r_{23}.$$

On the other hand, if we had assumed that $\rho_{34} < -\lambda'$, then the inequalities above would be reversed and, by Lemma 2.3, the configuration could not be realized geometrically.

(3) Let $\Gamma_1, \Gamma_2 > 0$ and $\Gamma_3, \Gamma_4 < 0$, and assume the vortices of the same sign are opposite one another. The signs of the four areas are then $A_1 > 0$, $A_2 > 0$, and $A_3 < 0$, $A_4 < 0$ (or vice versa). If we assume that $\rho_{34} > -\lambda'$, then we find from the equations in (9) that

$$\rho_{12}, \rho_{13}, \rho_{24}, \rho_{34}, \rho_{14}, \rho_{23} > -\lambda',$$

or

$$r_{12}, r_{13}, r_{24}, r_{34}, r_{14}, r_{23} < 1/\sqrt{-\lambda'}.$$

If $\rho_{34} < -\lambda'$, then the two previous inequalities are reversed. □

2.2 Symmetric Configurations

One immediate consequence of the Dziobek equations (8) and of Eq. (10) is that if two mutual distances containing a common vortex are equal (e.g., $r_{12} = r_{13}$), then the same equality of distances is true for the excluded vortex (here, $r_{24} = r_{34}$). Specifically, if i, j, k, l are distinct indices, then we have

$$r_{ij} = r_{ik} \quad \text{if and only if} \quad r_{lj} = r_{lk}. \tag{11}$$

This relation is independent of the vortex strengths although $\Gamma_j = \Gamma_k$ necessarily must follow. Any configuration satisfying Eq. (11) has an axis of symmetry containing vortices i and l, forming a kite configuration. It may be either convex or concave, but it cannot contain three vortices on a common line due to the equations in (7). Unlike the Newtonian four-body problem, since there is no restriction here on the signs

of the vortex strengths, it follows that *any* kite configuration has a corresponding set of vorticities that make it a relative equilibrium. Assuming the configuration is not an equilateral triangle with a vortex at the center, the vorticities are unique up to a common scaling factor and can be determined by the equations in (9).

Next we consider the case where two mutual distances without a common index are equal. In other words, suppose one of the following three equations holds:

$$r_{12} = r_{34}, \qquad r_{13} = r_{24}, \qquad r_{14} = r_{23}. \tag{12}$$

Then, it does not necessarily follow that another pair of mutual distances must be equal. However, if different pairs of vortices are assumed to be of equal strength, then we can conclude an additional symmetry. This fact will be important in Sect. 6.1 when verifying that symmetry is required in certain cases.

Lemma 2.5 (Symmetry Lemma) *Suppose that we have a strictly planar four-vortex relative equilibrium with $\Gamma_1 = \Gamma_2$, $\Gamma_3 = \Gamma_4$, and $\Gamma \neq 0$. Then,*

$$r_{13} = r_{24} \quad \textit{if and only if} \quad r_{14} = r_{23}. \tag{13}$$

If either equation in (13) holds, the configuration is convex and has either one or two axes of symmetry. In this case, the configuration is either an isosceles trapezoid with vortices 1 and 2 on one base, and 3 and 4 on the other, or it is a rhombus with vortices 1 and 2 opposite each other.

Proof Suppose that $r_{13} = r_{24}$. Then, by one of the equations in (7), $\Gamma_1 \Gamma_3 (r_{13}^{-2} + \lambda') = \Gamma_2 \Gamma_4 (r_{24}^{-2} + \lambda')$ implies that

$$A_1 A_3 = A_2 A_4. \tag{14}$$

We also have

$$A_1 + A_2 + A_3 + A_4 = 0 \tag{15}$$

since the A_i's are oriented areas. Solving Eq. (15) for A_4 and substituting into Eq. (14) yields the relation

$$(A_1 + A_2)(A_2 + A_3) = 0.$$

There are two possibilities. First, suppose that $A_1 = -A_2$. Then Eq. (15) immediately implies $A_4 = -A_3$. The configuration is convex due to the signs of the A_i's and must have the side containing vortices 1 and 2 parallel to the side containing vortices 3 and 4. Then, by another equation in (7), $\sigma A_1 A_4 = \sigma A_2 A_3$ implies that

$$\Gamma_1 \Gamma_4 (r_{14}^{-2} + \lambda') = \Gamma_2 \Gamma_3 (r_{23}^{-2} + \lambda').$$

Since $\Gamma_1 = \Gamma_2$ and $\Gamma_3 = \Gamma_4$, it follows that $r_{14} = r_{23}$. The configuration must be an isosceles trapezoid with congruent legs and diagonals, and base lengths given by r_{12} and r_{34}.

Next, suppose that $A_3 = -A_2$. Then, $A_4 = -A_1$ immediately follows from Eq. (15). The configuration is convex and must have the side containing vortices

1 and 4 parallel to the side containing vortices 2 and 3. Since $r_{13} = r_{24}$, the configuration is either an isosceles trapezoid with $r_{12} = r_{34}$ or is a parallelogram with $r_{14} = r_{23}$. For the isosceles trapezoid, the lengths r_{13} and r_{24} can correspond to either a pair of congruent diagonals or a pair of congruent legs, depending on the ordering of the vortices. However, for the parallelogram, these lengths must correspond to a pair of opposite sides as the diagonals do not have to be congruent.

It turns out that, in this case, the isosceles trapezoid is actually a square. To see this, by an equation in (7), $\sigma A_1 A_2 = \sigma A_3 A_4$ implies that

$$\Gamma_1^2 \left(r_{12}^{-2} + \lambda' \right) = \Gamma_3^2 \left(r_{34}^{-2} + \lambda' \right).$$

Thus, if $r_{12} = r_{34}$, then $\Gamma_1 = \Gamma_3$ and all vortex strengths are equal (the case $\Gamma_1 = -\Gamma_3$ is excluded since $\Gamma \neq 0$). In this case, it is straightforward to show that the isosceles trapezoid reduces to a square (see Sect. 7.1) and thus $r_{14} = r_{23}$.

In the case of the parallelogram, we have $A_1 = A_2 = -A_3 = -A_4$. By the equations in (7), this implies that $r_{13} = r_{14} = r_{23} = r_{24}$ and the configuration is a rhombus with vortices 1 and 2 opposite each other. This proves the forward implication.

The proof in the reverse direction is similar. If $r_{14} = r_{23}$, then we derive $A_1 A_4 = A_2 A_3$ from an equation in (7). Taken with Eq. (15), this yields

$$(A_1 + A_2)(A_1 + A_3) = 0.$$

As before, the case $A_1 = -A_2$ leads to an isosceles trapezoid with $r_{13} = r_{24}$ and base lengths given by r_{12} and r_{34}. The case $A_1 = -A_3$ leads to either the square or a rhombus configuration with vortices 1 and 2 across from each other. In either configuration we deduce that $r_{13} = r_{24}$. This completes the proof. □

Remark

1. Similar results exist if different pairs of vortices are assumed to be equal. For example, if $\Gamma_1 = \Gamma_3$ and $\Gamma_2 = \Gamma_4$, then $r_{12} = r_{34}$ if and only if $r_{14} = r_{23}$.
2. The result is also valid in the Newtonian four-body problem (and for other potentials of the same form) since it only depends on the geometry of the configuration and the inherent structure of the equations in (7).

2.3 The Albouy–Chenciner Equations

For the remainder of the paper (excluding the special cases discussed in Sect. 4), we will assume the equality of vortex strengths specified in Lemma 2.5. Specifically, we set $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$, treating m as a real parameter.

When $\Gamma \neq 0$, the equations for a relative equilibrium can be written in polynomial form as

$$f_{ij} = \sum_{k=1}^{n} \Gamma_k \left[S_{ik} \left(r_{jk}^2 - r_{ik}^2 - r_{ij}^2 \right) + S_{jk} \left(r_{ik}^2 - r_{jk}^2 - r_{ij}^2 \right) \right] = 0,$$

where $1 \leq i < j \leq n$ and the S_{ij} are given by

$$S_{ij} = \frac{1}{r_{ij}^2} + \lambda' \quad (i \neq j), \qquad S_{ii} = 0.$$

These very useful equations are due to Albouy and Chenciner ([1997](#)) (see also Hampton and Moeckel [2005](#) for a nice derivation). They form a polynomial system in the r_{ij} variables after clearing the denominators in the S_{ij} terms.

Since any relative equilibrium may be rescaled, we will impose the normalization $\lambda' = -1$ unless otherwise stated. This usually can be assumed without loss of generality. However, as explained in Sect. [2.1](#), for $m < 0$ the normalization $\lambda' = 1$ also needs to be considered. This case is discussed in Sects. [7.3](#) and [7.4](#). We denote the complete set of polynomial equations determined by $f_{ij} = 0$ as $\mathcal{F}$.

From the Albouy–Chenciner equations we can derive a more restrictive set of equations, namely

$$g_{ij} = \sum_{k=1}^{n} \Gamma_k S_{ik} \left(r_{jk}^2 - r_{ik}^2 - r_{ij}^2 \right) = 0.$$

Since $g_{ij} \neq g_{ji}$, these give 12 distinct equations. We denote the complete set of polynomial equations determined by $g_{ij} = 0$ as $\mathcal{G}$, and we will refer to them as the unsymmetrized Albouy–Chenciner equations.

The solutions of the Albouy–Chenciner equation give configurations of all dimensions, but, in the four-vortex problem, they can be specialized to the strictly planar case by adding the three Dziobek equations in ([8](#)). Introducing the variables $s_{ij} = r_{ij}^2$, these equations can be written as

$$h_{ijkl} = \left(s_{ij}^{-1} + \lambda' \right) \left(s_{kl}^{-1} + \lambda' \right) - \left(s_{ik}^{-1} + \lambda' \right) \left(s_{jl}^{-1} + \lambda' \right) = 0,$$

where i, j, k, and l are all distinct indices. We denote the set of Dziobek equations (with denominators cleared) as $\mathcal{H}$.

3 Algebraic Techniques

In this section we briefly describe several of our main algebraic techniques for analyzing solutions to our problem: Descartes' rule of signs, Sturm's theorem, elimination theory using Gröbner bases, a useful lemma to distinguish when the roots of a quartic are real or complex, and Mobius transformations.

3.1 Descartes' Rule of Signs and Sturm's Theorem

Descartes' rule of signs (see Uspensky [1948](#)) is a technique for determining an upper bound on the number of positive or negative roots of a polynomial. The rule states that the number of positive real roots of a polynomial with real coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

is either equal to the number of sign variations in the sequence of its coefficients $a_n, a_{n-1}, \ldots, a_0$, or is less than it by an even number. Multiple roots are counted separately. The same rule applies to find the number of negative roots once the signs of the coefficients of the odd-power terms have been flipped.

Sturm's theorem (see Uspensky 1948) provides a straightforward technique to find the exact number of real roots contained in an interval. Let $P(x)$ be a polynomial in x with real coefficients. The Sturm sequence of polynomials $(P_0(x), P_1(x), \ldots)$ is defined by

$$P_0(x) = P(x),$$
$$P_1(x) = P'(x),$$
$$P_n(x) = -\text{rem}(P_{n-2}, P_{n-1}), \quad n \geq 2,$$

where $\text{rem}(P_{n-2}, P_{n-1})$ denotes the remainder of the polynomial P_{n-2} upon division by the polynomial P_{n-1}. The sequence terminates once one of the P_i's is zero. We denote by $\text{var}_P(x_0)$ the number of sign changes in the sequence $P_0(x_0), P_1(x_0), \ldots$. Sturm's theorem states that if a and b are real numbers that are both not roots of $P(x)$, then the number of distinct real roots in (a, b) is equal to $|\text{var}_P(a) - \text{var}_P(b)|$.

3.2 Gröbner Bases and Elimination Theory

We mention briefly some elements from elimination theory and the theory of Gröbner bases that will prove useful in our analysis. For a more detailed exposition see Cox et al. (2007).

Let K be a field and consider the polynomial ring $K[x_1, \ldots, x_n]$ of polynomials in n variables over K. Let $f_1, \ldots, f_l$ be l polynomials in $K[x_1, \ldots, x_n]$ and consider the ideal $I = \langle f_1, \ldots, f_l \rangle$ generated by these polynomials. Denote $\mathbf{V}(I)$ as the affine variety of I.

Definition 3.1 An admissible order $>$ on $K[x_1, \ldots, x_n]$ is called a *k-elimination order* if

$$x_1^{a_1} \ldots x_n^{a_n} > x_{k+1}^{b_{k+1}} \ldots x_n^{b_n}$$

when $a_{i_0} > 0$ for some $i_0 \in \{1, \ldots, k\}$.

A lexicographical (lex) order is an example of k-elimination order for all k.

Definition 3.2 The *kth elimination ideal* I_k is the ideal of $K[x_{k+1}, \ldots, x_n]$ defined by

$$I_k = I \cap K[x_{k+1}, \ldots, x_n].$$

Gröbner bases provide a systematic way of finding elements of I_k using the proper term ordering.

Theorem 3.3 (The Elimination Theorem) *Let I be an ideal of $K[x_1, \ldots, x_n]$ and let G be a Gröbner basis of I with respect to a k-elimination order for k where $0 \leq k \leq n$. Then the set*

$$G_k = G \cap K[x_{k+1}, \ldots, x_n]$$

is a Gröbner basis of the kth elimination ideal I_k.

Gröbner bases also provide a method of determining when an element of $\mathbf{V}(I_k)$ (a partial solution) can be extended to a full solution in $\mathbf{V}(I)$. This can be achieved by repeatedly applying the following theorem.

Theorem 3.4 (The Extension Theorem) *Let K be an algebraically closed field and let I be some ideal in $K[x_1, \ldots, x_n]$. Let G_{k-1} be a lex Gröbner basis for the elimination ideal I_{k-1} and write each polynomial in G_{k-1} as*

$$g_i = h_i(x_{k+1}, \ldots x_n)x_k^{N_i} + \textit{terms where } x_k \textit{ has degree } < N_i,$$

where $N_i > 0$ and h_i is nonzero. Suppose that $(a_{k+1}, \ldots, a_n)$ is a partial solution in $\mathbf{V}(I_k)$ and that $h_i(a_{k+1}, \ldots, a_n) \neq 0$ for some index i. Then there exists $a_k \in K$ such that $(a_k, a_{k+1}, \ldots, a_n) \in \mathbf{V}(I_{k-1})$.

3.3 A Useful Lemma for Quartic Polynomials

The analysis of the collinear case, as well as some strictly planar cases, frequently involves solving a quartic equation whose coefficients are polynomials in m. We state here some useful results about quartic equations. Consider the general quartic polynomial $\zeta(x) = ax^4 + bx^3 + cx^2 + dx + e$, with coefficients in $\mathbb{R}$ and $a \neq 0$. We first remove the cubic term of ζ by the change of variables $x = y - \frac{b}{4a}$. This produces the polynomial $a\eta(y)$, where η is the *shifted quartic* $\eta(y) = y^4 + py^2 + qy + r$. The *discriminant* Δ of any polynomial is a positive constant times the square of the product of all possible differences of roots. The discriminant of ζ is equivalent to a^6 times the discriminant of η. For a general quartic, it is straightforward to check that if $\Delta > 0$, then the roots are either all real or all complex (two pairs of complex conjugates). If $\Delta < 0$, then there are two real roots and two complex roots. The roots are repeated if and only if $\Delta = 0$.

Let y_1, y_2, y_3, and y_4 be the four roots of $\eta(y)$. By construction, $y_1 + y_2 + y_3 + y_4 = 0$. It follows that

$$z_1 = -(y_1 + y_2)^2, \qquad z_2 = -(y_1 + y_3)^2, \qquad z_3 = -(y_1 + y_4)^2$$

are the roots of the *resolvent cubic* $\xi(z) = z^3 - 2pz^2 + (p^2 - 4r)z + q^2$. The discriminant of ξ is equivalent to the discriminant of η. When the discriminant is positive, the resolvent cubic is particularly useful for determining whether the roots of $\eta(y)$ are all real or all complex. Specifically, if the four roots of $\eta(y)$ are real, then z_1, z_2, and z_3 must be negative and real (or if $q = 0$, then $z_i = 0$ for precisely one i while the remaining z_i's are negative). On the other hand, if the four roots of $\eta(y)$ are complex, then one of the roots of $\xi(z)$ is less than or equal to zero, but two of the roots of $\xi(z)$ must be positive and real. These facts can easily be translated into conditions on the coefficients of the resolvent cubic.

Lemma 3.5 *Suppose that the discriminant of a quartic polynomial $\zeta(x)$ is positive and let $\eta(y) = y^4 + py^2 + qy + r$ be the shifted quartic related to ζ. Then the four distinct roots of ζ are real if and only if $p < 0$ and $p^2 - 4r > 0$.*

Proof Since the discriminant is positive, the roots of ζ are distinct and either all real or all complex. First suppose that the four roots of $\zeta(x)$ are real. Since $\eta(y)$ is obtained through a simple translation, it follows that the four roots of $\eta(y)$ are distinct and real. Then, the three real roots z_1, z_2, z_3 of the resolvent cubic $\xi(z)$ are negative with possibly one root equal to zero (if $q = 0$). This in turn implies that

$$z_1 + z_2 + z_3 = 2p < 0 \quad \text{and}$$
$$z_1 z_2 + z_1 z_3 + z_2 z_3 = p^2 - 4r > 0.$$

In the other direction, suppose that $p < 0$ and $p^2 - 4r > 0$. This implies that all the coefficients of $\xi(z)$ are positive, and by Descartes' rule of signs, $\xi(z)$ has no positive, real roots. It follows that the four roots of $\eta(y)$ cannot be complex. This shows that the roots of $\zeta(x)$ are real. $\square$

3.4 Möbius Transformations

Möbius transformations are a key tool for isolating roots of polynomials since they allow for a dramatic reduction on the number of variations of signs in the coefficients of a polynomial (Barros and Leandro 2011). Let $\mathcal{P}(x_1, \ldots, x_n)$ be a multivariate polynomial in n variables. We use Möbius transformations of the form

$$x_i = \frac{k_i^{(2)} y_i + k_i^{(1)}}{y_i + 1},$$

where $k_i^{(1)} < k_i^{(2)}$ and $i = 1, \ldots, n$, to obtain changes of variables for $\mathcal{P}(x_1, \ldots, x_n)$. The numerator of the rational function thus obtained is a multivariate polynomial in the variables $y_1, \ldots, y_n$ that restricted to the set $[0, \infty) \times \cdots \times [0, \infty)$ has the same number of roots as $\mathcal{P}(x_1, \ldots, x_n)$ restricted to $[k_1^{(1)}, k_1^{(2)}) \times \cdots \times [k_n^{(1)}, k_n^{(2)})$.

Möbius transformations can be used to determine partitions of the space into blocks so that, in each block, the coefficients of the polynomials under consideration have a very simple behavior with respect to variations of signs. Therefore, we can use such transformations, together with Descartes' rule of signs, to determine whether a polynomial has a root or does not have a root in a given region of space.

4 Special Cases

In this section, we use some results from Hampton and Moeckel (2009) to examine two special cases: equilibrium solutions and vanishing total vorticity ($\Gamma = 0$).

4.1 Symmetric Equilibria

Equilibria are solutions to Eq. (2) with $\lambda = 0$. If we choose coordinates so that the fourth vortex is at the origin, and the third vortex is at $(1, 0)$, then the other two

vortices in a four-vortex equilibrium must be located at

$$
\begin{aligned}
x_1 &= \frac{1}{2(\Gamma_2 + \Gamma_3 + \Gamma_4)}(2\Gamma_4 + \Gamma_2, \pm\sqrt{3}\Gamma_2), \\
x_2 &= \frac{1}{2(\Gamma_1 + \Gamma_3 + \Gamma_4)}(2\Gamma_4 + \Gamma_1, \mp\sqrt{3}\Gamma_1),
\end{aligned}
\tag{16}
$$

where the sign chosen for x_2 is the opposite of that for x_1 (see Hampton and Moeckel 2009 for details).

According to Eq. (4), for an equilibrium to exist the vorticities must satisfy $L = 0$. This is impossible if all the vorticities are equal. If three vorticities are equal, for example with $\Gamma_1 = \Gamma_2 = \Gamma_3 = m$, then $\Gamma_4 = -m$. We can rescale so that $m = 1$ without loss of generality. Using (16) it is easy to see that the only equilibria are equilateral triangles with the opposing vortex located in the center.

Now consider equilibria when $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. Then, $L = 0$ implies $m = -2 \pm \sqrt{3}$. Since we are restricting to $m \in (-1, 1]$, we have $m = -2 + \sqrt{3} \approx -0.2679$. In this case, the equilibria are rhombi with vortices 1 and 2 opposite each other, and $\frac{r_{34}}{r_{12}} = 2 - \sqrt{3} = -m$. They possess an interesting geometric property: the ratio of the smaller diagonal to the larger diagonal equals the square of the common side length. These rhombi are members of the rhombus B family described in Sect. 7.4.

Suppose we instead specify the symmetry of the configuration. From (16) it is immediate that there are no collinear equilibria for nonzero vorticities. There are also no isosceles trapezoid equilibria. To see this, consider a trapezoid with $r_{14} = r_{23}$ and $r_{13} = r_{24}$. Then (16) implies that

$$
\frac{\Gamma_2}{\Gamma_2 + \Gamma_3 + \Gamma_4} = \frac{-\Gamma_1}{\Gamma_1 + \Gamma_3 + \Gamma_4},
$$

but there are no real nonzero vorticities satisfying this equation and $L = 0$.

Finally, there is the case in which the configuration is a kite (either concave or convex). We choose vortices 1 and 2 to be on an axis of symmetry, so $r_{13} = r_{14}$ and $r_{23} = r_{24}$. Then Eq. (16) implies that $\Gamma_3 = \Gamma_4$. If we choose any $\Gamma_2 \neq -2\Gamma_4$ with

$$
\Gamma_2 = -\frac{2\Gamma_1\Gamma_4 + \Gamma_4^2}{\Gamma_1 + 2\Gamma_4},
$$

then $L = 0$ is satisfied and there is a kite equilibrium given by (16). If we fix $\Gamma_4 = 1$, then as $\Gamma_2 \to -2$, $\Gamma_1 \to \infty$ and the configuration of vortices 2, 3, and 4 approaches an equilateral triangle while the ordinate of vortex 1 heads off to $\pm\infty$.

4.2 Vanishing Total Vorticity

When the total vorticity $\Gamma = 0$, the analysis of stationary vortex configurations (i.e., those that do not change their shape) requires equations adapted to this special case. For four vorticities we simply apply some of the results from Hampton and Moeckel (2009).

If three of the vorticities are equal and $\Gamma = 0$, we can consider without loss of generality the case $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$ and $\Gamma_4 = -3$. It is interesting that there are six asymmetric configurations which rigidly translate (three pairs of configurations, within a pair the configurations are reflections of one another), but no symmetric rigidly translating solutions. These configurations can be obtained directly from the equations in Hampton and Moeckel (2009).

For relative equilibria we use the equations

$$S_1 = S_2 = S_3 = S_4 = s_0$$

and

$$\frac{1}{s_{12}} + \frac{1}{s_{34}} = \frac{1}{s_{13}} + \frac{1}{s_{24}} = \frac{1}{s_{14}} + \frac{1}{s_{23}}, \tag{17}$$

where

$$S_i = \Gamma_j s_{ij} + \Gamma_k s_{ik} + \Gamma_l s_{il}, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}$$

and s_0 is an auxiliary variable. Recall that $s_{ij} = r_{ij}^2$. We clear denominators in the equations from (17) to get a polynomial system. Using a Gröbner basis to eliminate s_0, we find that for $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$, and $\Gamma_4 = -3$ there are two types of symmetric relative equilibria. The first is the equilateral triangle with vortex 4 at its center. The second type is a concave kite with the three equal vorticities on the exterior isosceles triangle. If we scale the exterior triangle so that its longest side is length 1, then the base is length $\sqrt{\sqrt{3} - 5}$, and the other sides of the interior triangle containing vortex 4 are length $\sqrt{-1 + \sqrt{3}}$.

If two pairs of vorticities are equal and $\Gamma = 0$, then we have $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = -1$. In this case there are no rigidly translating solutions. This is somewhat surprising since each pair of opposing vortices would rigidly translate if unperturbed. Using the same equations as we did for the case where three vorticities are equal, we find that there are two relative equilibria, each of which forms a rhombus. These appear in the two families of rhombi described in Sect. 7.4, with $\frac{s_{34}}{s_{12}} = 3 \pm 2\sqrt{2}$.

5 Collinear Relative Equilibria

Collinear relative equilibria of the four-vortex problem can be studied directly from Eq. (2) since in this case it reduces to

$$-\lambda(x_i - c) = \sum_{\substack{j \neq i}}^{n} \frac{\Gamma_j}{x_j - x_i} \quad \text{for each } i \in \{1, 2, 3, 4\}, \tag{18}$$

where $c, x_i \in \mathbb{R}$ rather than $\mathbb{R}^2$. Clearing denominators from these equations yields a polynomial system. Rather than fix λ or c, we use the homogeneity and translation invariance of the system and set $x_3 = -1$ and $x_4 = 1$. As specified earlier, we set $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m \in (-1, 1]$, treating m as a parameter.

5.1 Symmetric Solutions

Given our setup, symmetric configurations correspond to solutions where either $x_1 = -x_2$ or $r_{12} = 2$. Both cases are simple enough to analyze using Gröbner bases.

Case 1: $x_1 = -x_2$

In this case the center of vorticity c is at the origin. The solutions are the roots of an even quartic polynomial in x_1, given by $mx_1^4 - 5(m+1)x_1^2 + 1$. The discriminant of this quartic changes sign at $m = 0$ and there are four real roots for $m > 0$ but only two when $m < 0$. The result is that, for any $m \in (-1, 1]$, there is a collinear relative equilibrium with

$$x_1 = -x_2 = \pm\sqrt{\frac{5m + 5 - \sqrt{25m^2 + 46m + 25}}{2m}} \tag{19}$$

(with $x_1 = -x_2 = \pm 1/\sqrt{5}$ when $m = 0$). This solution has vortices x_1 and x_2 symmetrically located between vortices x_3 and x_4, and as $m \to -1^+$, the inner vortices approach the outer vortices (with collision at $m = -1$.) For $m \in (0, 1]$, $|x_1|$ decreases monotonically in m from $1/\sqrt{5} \approx 0.4472$ to $\sqrt{3} - \sqrt{2} \approx 0.3178$, while for $m \in (-1, 0)$, $|x_1|$ decreases monotonically in m from 1 to $1/\sqrt{5}$.

There is an additional collinear relative equilibrium if $m \in (0, 1]$ given by

$$x_1 = -x_2 = \pm\sqrt{\frac{5m + 5 + \sqrt{25m^2 + 46m + 25}}{2m}}.$$

In this case, the vortices x_1 and x_2 are symmetrically located outside vortices x_3 and x_4, and their positions approach $\pm\infty$ as $m \to 0^+$. The value of $|x_1|$ decreases monotonically in m from ∞ to $\sqrt{3} + \sqrt{2} \approx 3.1462$.

Case 2: $r_{12} = 2$

In this case the center of vorticity c is not necessarily located at the origin; however, the configuration will be symmetric about some fixed point equidistant from both the inner and the outer pairs of vortices. Since $|x_1 - x_2| = 2$ and $x_1 = -x_2$ together imply $|x_2| = 1$ (collision), the two cases are distinct. If $m \neq 1$, a solution for case 2 will have the inner and outer pair of vortices having different circulations. It turns out that this case is impossible when $m \neq 1$.

To see this, we consider the polynomials obtained from system (18) along with $x_1 - x_2 - 2$ and $u(x_1 + x_2) - 1$ (to eliminate solutions from Case 1). Computing a lex Gröbner basis for this set of polynomials quickly yields $m = 1$. The same result is obtained when using $x_1 - x_2 + 2$. Moreover, in each computation, a fourth-degree polynomial in x_2 is obtained that provides the exact solution for the special case $m = 1$.

In sum, there are no solutions with $r_{12} = 2$ unless $m = 1$. When $m = 1$, there are eight solutions with $r_{12} = 2$ given by $(x_1, x_2) =$

$$(1 \pm \sqrt{2} \pm \sqrt{6}, -1 \pm \sqrt{2} \pm \sqrt{6}) \quad \text{and} \quad (-1 \pm \sqrt{2} \pm \sqrt{6}, 1 \pm \sqrt{2} \pm \sqrt{6}), \tag{20}$$

where the signs in a particular ordered pair are chosen to have the same pattern in each coordinate (e.g., $(1 + \sqrt{2} - \sqrt{6}, -1 + \sqrt{2} - \sqrt{6})$). It is straightforward to check that

each of these 8 solutions is a scaled version of a solution found in Case 1. Specifically, for each of the solutions listed in (20) as well as the solutions obtained in Case 1 for $m = 1$, there is a rescaling, translation, and relabeling of the vortices that maps the solution onto

$$x_3 = -1, \qquad x_2 = -\sqrt{3} + \sqrt{2}, \qquad x_1 = \sqrt{3} - \sqrt{2}, \qquad x_4 = 1, \qquad (21)$$

a solution obtained from Eq. (19) when $m = 1$. We note that solution (21) can be rescaled to coincide with the roots of the Hermite polynomial H_4, as expected (see Aref 2007a).

5.2 Asymmetric Solutions

To locate any asymmetric solutions, we introduce the variables u and v along with the equations

$$u(x_1 + x_2) - 1 \quad \text{and} \quad v(x_1 - x_2) - 1.$$

Adding these two equations to the original polynomial system obtained from (18), we compute a Gröbner basis G_{col} with respect to the lex order where $c > \lambda > u > v > x_1 > x_2 > m$. This basis has 15 elements, the first of which is an even, eighth-degree polynomial in x_2 with coefficients in m, and a basis for the elimination ideal $I_5 = I \cap \mathbb{C}[x_2, m]$. Introducing the variable $w = x_2^2$, this polynomial is given by

$$\zeta(w) = m^2(m + 2)(1 + 2m)^2 w^4 - 4m(15m^4 + 61m^3 + 91m^2 + 61m + 15)w^3$$
$$+ (300m^5 + 1508m^4 + 2910m^3 + 2696m^2 + 1188m + 200)w^2$$
$$- 4(5m + 4)(25m^4 + 127m^3 + 231m^2 + 175m + 45)w + (m + 2)^3.$$

If we eliminate x_2 instead of x_1, the same polynomial is obtained with $w = x_1^2$. The discriminant of ζ is

$$1048576(m + 2)^2(m + 1)^6(1 + 2m)(25m^2 + 58m + 25)^3 m^2 (q_u(m))^2,$$

where $q_u(m) = 2m^5 - 16m^4 - 96m^3 - 162m^2 - 108m - 25$. For $m \in (0, 1]$, the discriminant is strictly positive. However, on the interval $[-1, 0]$, the discriminant vanishes at six different m-values. In increasing order, these values are

$$-1, m_0 \approx -0.6833, m_1 \approx -0.6066, m_2 \approx -0.5721, -1/2, 0.$$

The values m_0 and m_1 are roots of the quintic polynomial q_u, and $m_2 = (-29 + 6\sqrt{6})/25$ is the largest root of the quadratic $25m^2 + 58m + 25$. For $m = -1$, ζ has a repeated root at 1 of multiplicity four. At $m = -1/2$, ζ becomes a cubic polynomial with roots -3 and 1 (multiplicity 2). Thus, the values $m = -1$ and $m = -1/2$ have no physical solutions (just collisions). For $m = 0$, the quartic reduces to a quadratic, and we have two possible solutions at $w = (9 \pm 4\sqrt{5})/5$ which correspond to limiting configurations as $m \to 0^+$. The other three values where the discriminant vanishes also have no physical solutions, as we explain below.

Lemma 5.1 *The quartic polynomial $\zeta(w)$ has precisely four positive real roots for $m \in (0, 1]$ and two positive real roots for $m \in (-1/2, 0)$. For $m \in (-1, -1/2)$, there are no positive roots except when $m = m_0$, where there is one positive, repeated root w_0 of multiplicity two.*

Proof Using Lemma 3.5, let η be the shifted quartic of $\zeta(w)$. From η, the key quantities p and $p^2 - 4r$ are found to be

$$p = -\frac{2(25m^2 + 58m + 25)(3m^4 + 14m^3 + 45m^2 + 54m + 19)(m + 1)^2}{m^2(m + 2)^2(1 + 2m)^4}$$

and

$$p^2 - 4r = \frac{16(25m^2 + 58m + 25)(m + 1)^2 t(m)}{m^4(m + 2)^4(1 + 2m)^8},$$

where

$$t(m) = 75m^{12} + 896m^{11} + 5528m^{10} + 23492m^9 + 77272m^8 + 197816m^7$$
$$+ 376194m^6 + 509968m^5 + 478976m^4 + 302388m^3 + 121964m^2$$
$$+ 28320m + 2875.$$

For $m \in (0, 1]$, since the discriminant is positive, $p < 0$ and $p^2 - 4r > 0$, Lemma 3.5 applies and the roots of $\zeta(w)$ are all real. The fact that they are all positive follows from Descartes' rule of signs since $\zeta(w)$ has four sign changes while $\zeta(-w)$ has none. For $m \in (-1/2, 0)$, the discriminant remains positive and using Sturm's theorem, one can show that $p < 0$ and $p^2 - 4r > 0$ both continue to hold. It follows that $\zeta(w)$ has four real roots in this case as well. However, two of the roots are positive and two are negative since both $\zeta(w)$ and $\zeta(-w)$ each have two sign changes.

For the case $m \in (m_2, -1/2)$, the discriminant is negative and thus there are precisely two real roots. To show that the real roots are both negative, we note that at $m = m_2$, the quartic ζ has a repeated root of multiplicity four at $w_2 = -(11 + 4\sqrt{6})/5 < 0$. We then can check that $\zeta(w_2)$ as a function of m is strictly negative for $m \in (m_2, -1/2)$. Since the leading coefficient of ζ is positive and since $\zeta(0) = (m + 2)^3 > 0$, it follows that both real roots must be negative in this case. There are no positive roots at $m = m_2$ since $w_2 < 0$ is the only root.

For the interval $m \in (-1, m_2)$, the discriminant is positive except when it vanishes at m_0 and m_1. However, either $p > 0$ or $p^2 - 4r < 0$ (or both) for m-values on this interval. Consequently, by Lemma 3.5, if $m \in (-1, m_2) - \{m_0, m_1\}$, then ζ has four complex roots.

It remains to check the two cases $m = m_0$ and $m = m_1$, where the discriminant of ζ vanishes. Using the resolvent cubic, it is possible to show that, for each case, a pair of complex conjugate roots meet on the real axis to form a real root of multiplicity two. At $m = m_1$, the derivative of $\zeta(w)$ is a cubic with only one real negative root. Consequently, the repeated real root in this case is negative. For $m = m_0$, the sign of the coefficient of the linear term for the shifted quartic $\eta(w)$ is negative, which

implies that the repeated real root w_0 is positive. Using resultants, it can be shown that w_0 is the largest root of the quintic $5w^5 - 53w^4 + 98w^3 + 198w^2 + 9w - 1$, or $w_0 \approx 6.9632775$. □

While it is interesting that the special parameter value $m = m_0$ has a unique positive solution, it cannot lead to a physical solution of the problem because x_1^2 necessarily has to be a *different* root of ζ (since we are excluding the symmetric solutions). But the other roots of ζ are complex. The fact that x_1 is complex can also be confirmed by solving one of the polynomials in the Gröbner basis G_{col} for x_1 when $m = m_0$ and $x_2 = \sqrt{w_0}$.

We summarize our findings for the collinear case in the following theorem.

Theorem 5.2 *For the case $m \in (0, 1]$, there are* 12 *collinear solutions, one for each possible ordering of the vortices. If $m = 1$, all solutions are symmetric and geometrically equivalent to the same configuration; otherwise, there are four symmetric solutions and eight asymmetric solutions. For $m \in (-1/2, 0)$, there are a total of six solutions, two symmetric and four asymmetric, while for $m \in (-1, -1/2]$, there are only two symmetric solutions and no asymmetric solutions.*

Proof The second polynomial in the Gröbner basis G_{col}, denoted g_2, is in the elimination ideal $I_4 = I \cap \mathbb{C}[x_1, x_2, m]$ and is linear in the variable x_1. The coefficient of x_1 is found to be $8(2m + 1)(m + 2)^2 \cdot q_u(m)$, which is nonzero for $m \in (-1/2, 1]$. By the Extension Theorem, we can extend any solution (x_2, m) of $\zeta = 0$ to a partial solution (x_1, x_2, m) in $\mathbf{V}(I_4)$. Moreover, since g_2 is linear in x_1, there is at most one such solution, and x_1 must be real.

Next, we check that the value of x_1 is distinct from the value of x_2 (excluding collision) and $-x_2$ (excluding the symmetric solutions). This is accomplished by substituting $x_1 = x_2$ into g_2 and then computing the resultant of this polynomial with $\zeta(x_2, m)$. A polynomial in m is thus obtained and it is easily checked that this polynomial has no roots for $m \in (-1/2, 1]$. Therefore, the value of x_1 obtained by solving $g_2 = 0$ is distinct from x_2. A similar calculation, using the substitution $x_1 = -x_2$ in g_2, shows that $x_1 \neq -x_2$ as well.

At this point, the Extension Theorem can be applied four more times using four of the basis polynomials in G_{col} that are linear in the variables v, u, λ, and c, respectively, each having nonzero leading coefficients. Thus, fixing an $m \in (-1/2, 1]$, for each positive root of $\zeta(w)$, we obtain two possible values of x_2, each of which extends uniquely to a full, real, asymmetric solution of our problem. The precise count on the number of solutions for each case then follows directly from Lemma 5.1 and Sect. 5.1.

The fact that there is one collinear relative equilibrium for each ordering of the vortices when $m > 0$ follows from a straightforward generalization of a well-known result in the Newtonian n-body problem due to Moulton (1910). In brief, for each of the $n!$ connected components of the phase space for the collinear n-vortex problem, there is a unique minimum of H restricted to the ellipsoid $I = I_0$ (see p. 33 of Meyer et al. 2009 or Sect. 6.1 of O'Neil 1987 for details). Each such minimum is a collinear relative equilibrium. Identifying solutions equivalent under a 180° rotation of the plane gives a final count of $n!/2$. □

Theorem 5.3 *When $m \in (-1, 0)$, the signs of the vorticities in a collinear relative equilibrium must be arranged as $+--+$ (symmetric case only), or as $+-+-$ or $-+-+$ (when asymmetric solutions exist). Therefore, when $m < 0$, it is not possible to have a collinear solution where both pairs of vortices with the same strength are adjacent to each other.*

Proof The symmetric case has already been analyzed at the start of this section. For the asymmetric case, we note that

$$\zeta(1) = -128(1 + 2m)(m + 2)^2(m + 1)^2$$

is strictly negative when $m \in (-1/2, 0)$. We also note that $\zeta(0) > 0$ and the leading coefficient of $\zeta(w)$ is always positive. Since ζ has only two positive roots for $m \in (-1/2, 0)$, it follows that one root is less than one and the other is larger than one. (The two roots approach one as $m \to -1/2^+$.) The values of x_1^2 and x_2^2 must each be roots of $\zeta(w)$. Because we are only considering asymmetric solutions, we have $|x_1| < 1$ and $|x_2| > 1$ or vice versa. Therefore, when $m \in (-1/2, 0)$, the only possible asymmetric orderings of vortices have signs arranged as $+-+-$ or $-+-+$. □

6 Asymmetric Relative Equilibria

6.1 Eliminating Symmetric Solutions

In this section we study strictly planar relative equilibria that do not have a line of symmetry. A major result proved here is that any convex solution with $m > 0$ or any concave solution with $m < 0$ must contain a line of symmetry. We prove this by saturating the Gröbner basis in order to eliminate any symmetric solutions, and then showing the resulting system has no real solutions.

Let $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ be the Albouy–Chenciner and the unsymmetrized Albouy–Chenciner equations in terms of $s_{ij} = r_{ij}^2$, respectively, with $\lambda' = -1$, $\Gamma_1 = \Gamma_2 = 1$, and $\Gamma_3 = \Gamma_4 = m \neq -1$. Let $\tilde{e}_{\text{CM}}$ be the Cayley–Menger determinant written in terms of $s_{ij} = r_{ij}^2$ and let $\widetilde{\mathcal{H}}$ be the Dziobek equations with $\lambda' = -1$. The polynomials $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \tilde{e}_{\text{CM}}$, and $\widetilde{\mathcal{H}}$ belong to the polynomial ring $\mathbb{C}[m, s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}]$.

We begin by finding a Gröbner basis for the ideal $I_s = \langle \widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \tilde{e}_{\text{CM}}, \widetilde{\mathcal{H}} \rangle$. In order to accomplish this we first find a Gröbner basis G_{J_s} for the ideal $J_s = \langle \widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \tilde{e}_{\text{CM}} \rangle$ and then we compute a Gröbner basis G_{I_s} for $I_s = \langle G_{J_s}, \widetilde{\mathcal{H}} \rangle$. At this stage we saturate with respect to the variables s_{13}, s_{14}, and s_{24}, to eliminate possible solutions where one of the mutual distances has zero length. We also saturate with respect to $(s_{13} - s_{24})$, $(s_{14} - s_{23})$, $(s_{13} - s_{14})$, $(s_{23} - s_{24})$, $(s_{13} - s_{23})$, and $(s_{14} - s_{24})$. Due to Lemma 2.5 and Eq. (11), saturating with respect to these differences is equivalent to eliminating any symmetric solutions. We denote the resulting Gröbner basis as $\widetilde{G}_{I_s}$.

Computing $\widetilde{G}_{I_s}$ with respect to an elimination order that eliminates all the variables except s_{12} and s_{34} yields the following system of two equations in two unknowns:

$$s_{34}m + s_{12} - m - 1 = 0, \qquad s_{12}^2 - 2s_{12}s_{34} + s_{34}^2 - 1 = 0.$$

This system has the two solutions

$$(s_{12}, s_{34}) = \left(\frac{2m+1}{m+1}, \frac{m}{m+1}\right), \qquad (s_{12}, s_{34}) = \left(\frac{1}{m+1}, \frac{m+2}{m+1}\right), \qquad (22)$$

where one solution is sent to the other one by the transformation $m \to \frac{1}{m}$ and $s_{12} \leftrightarrow s_{34}$. If we compute the Gröbner basis of $\tilde{G}_{I_s}$ with respect to an elimination order that eliminates all but one $s_{ij} = x$, we obtain a polynomial that is the product of the two polynomials

$$\begin{aligned} p_1 = {}& 4(m^2 + 2m + 1)x^4 - 4(5m^2 + 8m + 3)x^3 + 2(16m^2 + 21m + 7)x^2 \\ & - 2(10m^2 + 11m + 3)x + 4m^2 + 4m + 1, \quad \text{and} \\ p_2 = {}& 4(m^2 + 2m + 1)x^4 - 4(3m^2 + 8m + 5)x^3 + 2(7m^2 + 21m + 16)x^2 \\ & - 2(3m^2 + 11m + 10)x + m^2 + 4m + 4, \end{aligned}$$

where x is one of s_{13}, s_{14}, s_{23} or s_{24}. For the case when s_{12} or s_{34} is the only variable not eliminated, we obtain the products

$$(ms_{12} + s_{12} - 1)(ms_{12} + s_{12} - 2m - 1), \quad \text{or}$$

$$(ms_{34} + s_{34} - m)(ms_{34} + s_{34} - m - 2),$$

respectively. All the Gröbner basis computations were performed using SINGULAR (Decker et al. 2011) and Sage (Stein et al. 2011).

Analyzing the polynomials p_1 and p_2 (which we do in the next section), one can prove the following:

Lemma 6.1 *For any $m \in (-1, 1)$, the polynomial p_1 has no real positive roots. For any $m \in (-1, 1)$, the polynomial p_2 has four positive distinct real roots except for $m = 0$, where one of the roots is repeated (at 1). If $m \in [0, 1]$, each root of p_2 lies in one of the intervals $J_1 = [0, \frac{1}{2}]$, $J_2 = [\frac{1}{2}, 1]$, $J_3 = [1, \frac{3}{2}]$, and $J_4 = [\frac{3}{2}, 5]$. If $m \in (-1, 0)$, each root of p_2 lies in one of the intervals $K_1 = [0, \frac{1}{2}]$, $K_2 = [\frac{1}{2}, 1]$, $K_3 = [1, \frac{m+2}{m+1}]$ and $K_4 = [\frac{m+2}{m+1}, \infty]$.*

Using Lemma 6.1, we obtain the following fundamental result:

Theorem 6.2 *Consider the four-vortex problem with vorticities $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$.*

1. *For any $m > 0$, every convex relative equilibrium has a line of symmetry, and the only possible strictly planar asymmetric configurations are concave with the two vortices of smaller strength lying on the exterior triangle. If $m = 1$, all solutions contain a line of symmetry.*
2. *For any $m < 0$, every concave relative equilibrium has a line of symmetry, and the only possible strictly planar asymmetric configurations are convex with equal-strength vortices necessarily adjacent.*

Proof First note that the case $\lambda' > 0$ is necessarily excluded in the hypotheses of the theorem. If $m > 0$, $\lambda' = -\lambda/\Gamma < 0$ is guaranteed. If $m < 0$ and $\lambda' > 0$, then the configuration must be convex. Hence, we can assume that $\lambda' = -1$.

If we compute a Gröbner basis of $\tilde{G}_{I_s}$ that eliminates all the s_{ij} variables except s_{12} and s_{13}, we obtain several polynomials including

$$p_3 = (ms_{12} - 2m + s_{12} - 1)(ms_{12} + s_{12} - 1)$$

and

$$
\begin{aligned}
p_4 = {}& -8ms_{13}^4 + 8(ms_{12} + 3m - s_{12} + 1)s_{13}^3 - 2\left(9ms_{12} - 4s_{12}^2 + 14m - s_{12} + 5\right)s_{13}^2 \\
& + 2\left(7ms_{12} - 4s_{12}^2 + 6m + s_{12} + 3\right)s_{13} - 3ms_{12} + 2s_{12}^2 - 2m - s_{12} - 1. \quad (23)
\end{aligned}
$$

The roots of p_3 are $\sigma_1 = \frac{2m+1}{m+1}$ and $\sigma_2 = \frac{1}{m+1}$. Substituting σ_1 into p_4, clearing denominators, and rescaling by a constant gives the value $p_1(s_{13})$, which is nonzero for $m \in (-1, 1)$ by the first statement in Lemma 6.1. Substituting σ_2 into p_4, clearing denominators, and rescaling by a constant gives the value $p_2(s_{13})$. For the case $m \in (-1, 1)$, p_2 has four distinct positive real roots. Thus, in order to have a geometrically realizable solution, we must have $s_{12} = 1/(m + 1)$ and s_{13} must be one of the four positive roots of p_2. From Eq. (22), we also have $s_{34} = (m + 2)/(m + 1)$.

Since $\lambda' = -1$, we have $1/\sqrt{-\lambda'} = 1$ in Propositions 2.2 and 2.4. If $m \in (0, 1)$, then $s_{12} < 1$ and $s_{34} > 1$. Since we have saturated with respect to all possible differences of the remaining four s_{ij} variables, the values of $s_{13}, s_{14}, s_{23}, s_{24}$ must all be *distinct* roots of p_2. By Lemma 6.1, two of these values are less than one, and two are greater than one. By Proposition 2.4 part 1, the configuration cannot be convex. Since we have saturated the Gröbner basis to eliminate symmetric solutions, it follows that, for $m \in (0, 1)$, the only strictly planar, asymmetric configurations are concave. By Proposition 2.2 part 1., vertices 3 and 4 (with equal strength m) lie on the outer triangle, while either vertex 1 or 2 can lie in the interior of the concave configuration. (The situation is reversed under the transformation $m \mapsto \frac{1}{m}$.)

For the special case $m = 1$, we have $s_{12} = 3/2$ and $s_{34} = 1/2$ or vice versa. If $s_{12} = 3/2 = \sigma_1$, then substitution into p_4 yields the result that s_{13} is a root of p_1. But the only real roots of p_1 when $m = 1$ are $1/2$ and $3/2$. Thus, we either have $s_{12} = s_{13}$ or $s_{34} = s_{13}$ and by Eq. (11), the configuration is necessarily a kite (a symmetric configuration). If $s_{12} = 1/2 = \sigma_2$, a similar argument with p_2 replacing p_1 also yields a kite configuration. This completes the proof of part 1 of the theorem.

If $m \in (-1, 0)$, we have $s_{12} > 1$, $s_{34} > 1$. Taken together with Lemma 6.1, this implies that four mutual distances are greater than one and two are less than one. Therefore, from Propositions 2.2 and 2.4, it follows that the only real, positive solutions to the system of equations given by $\tilde{G}_{I_s}$ correspond to convex asymmetric configurations where the equal-strength vortices are adjacent. This proves part 2 of the theorem. $\qquad \square$

Remark The fact that all solutions are symmetric in the case of four equal-strength vortices was first proved by Albouy (see Sect. 5 in Albouy 1996).

6.2 Proof of Lemma 6.1

In this section we analyze the polynomials p_1 and p_2 and prove Lemma 6.1. For p_1, we use Lemma 3.5, while for p_2, we make appropriate choices of Möbius transformations.

Both p_1 and p_2 are polynomials of degree four. Moreover, if $m > 0$, their coefficients have four sign changes, and thus they have either four, two or zero positive real roots, according to Descartes' rule of signs. The roots of the polynomials can be obtained using Ferrari's formula for quartic equations (Cardano 1545). However, we can understand a great deal about the solutions by using the resolvent cubic and Möbius transformations.

One important observation is that if we make the change $m \mapsto \frac{1}{m}$ in the polynomial p_1, and clear the denominators, we obtain p_2. In particular, this means that the roots of p_2 for $m \in (0, 1)$ are equivalent to the roots of p_1 for $m \in (1, \infty)$. Hence, it suffices to study the polynomials for $|m| < 1$.

Discriminants of p_1 and p_2   The discriminant of p_1 is $256(m + 1)^4(5m + 3)^2(2m + 1)^2(m - 1)^2$, which is strictly positive except for the m-values $-1, -3/5, -1/2$, and 1 where it vanishes. At $m = -1$, p_1 reduces to a quadratic polynomial with repeated roots at $x = 1/2$. For $m = -3/5$, p_1 has only complex roots, while for $m = -1/2$, p_1 has a double root at $x = 0$. At $m = 1$, there are double real roots at $x = 1/2$ and $x = 3/2$.

The discriminant for p_2 is $256(m + 2)^2(3m + 5)^2(m + 1)^4 m^2(m - 1)^2$ which is strictly positive except for the m-values $-2, -5/3, -1, 0$, and 1 where it vanishes. Focusing on the values between -1 and 1, we find that, at $m = -1$, p_2 reduces to a quadratic polynomial with repeated roots at $x = 1/2$. For $m = 0$, p_2 has a double root at 1 and two other roots at $(3 \pm \sqrt{5})/2$. At $m = 1$, p_2 has double real roots at $x = 1/2$ and $x = 3/2$. For all values of $m \in (-1, 1)$, except for $m = 0$, the four roots of p_2 are distinct.

Polynomial p_1   After applying the shift $x \mapsto x + (m + 1)(5m + 3)/(4(m + 1)^2)$ to remove the cubic term in p_1, we compute the key coefficients of the shifted quartic to be

$$p = -\frac{11m^2 + 6m - 1}{8(m + 1)^2}, \quad \text{and}$$

$$p^2 - 4r = \frac{(m - 1)(5m + 3)(7m + 3)}{16(m + 1)^3}.$$

It is straightforward to check that either $p > 0$ or $p^2 - 4r < 0$ (or both), for each m-value in $(-1, 1)$. Using Lemma 3.5, it follows that the roots of p_1 are all complex for $m \in (-1, 1)$, except when $m = -1/2$, where 0 is a double root.

Polynomial p_2: The case $m \in [0, 1]$ Consider the intervals $J_1 = [0, \frac{1}{2}], J_2 = [\frac{1}{2}, 1], J_3 = [1, \frac{3}{2}]$, and $J_4 = [\frac{3}{2}, 5]$. We show, that for each $m \in [0, 1]$, p_2 has a root in each of these intervals. The result follows from direct computation for $m = 0$ and $m = 1$, noting that some of the roots are a shared endpoint of adjacent intervals.

x-interval	Möbius Transformation	
J_1	$x = \frac{1}{2}\frac{u}{u+1},$	$m = \frac{\alpha}{\alpha+1}$
J_2	$x = \frac{u+\frac{1}{2}}{u+1},$	$m = \frac{\alpha}{\alpha+1}$
J_3	$x = \frac{\frac{3}{2}u+1}{u+1},$	$m = \frac{\alpha}{\alpha+1}$
J_4	$x = \frac{5u+\frac{3}{2}}{u+1},$	$m = \frac{\alpha}{\alpha+1}$

Using the Möbius transformations given in the table above, after clearing the denominators, we obtain the following four polynomials:

$$P_{J_1} = -(2\alpha+1)u^4 - 2(2\alpha+1)u^3 + 2\big(8\alpha^2+11\alpha+4\big)u^2 + 4\big(12\alpha^2+17\alpha+6\big)u$$
$$+ 36\alpha^2 + 48\alpha + 16,$$

$$P_{J_2} = 4\alpha^2 u^4 + 4\big(4\alpha^2+\alpha\big)u^3 + 2\big(8\alpha^2-\alpha-2\big)u^2 - 6(2\alpha+1)u - 2\alpha - 1,$$

$$P_{J_3} = -(18\alpha+5)u^4 - 2(22\alpha+5)u^3 + 2\big(8\alpha^2-13\alpha-2\big)u^2 + 4\alpha(4\alpha-1)u + 4\alpha^2,$$

$$P_{J_4} = 4\big(3969\alpha^2+3352\alpha+704\big)u^4 + 4\big(1764\alpha^2+815\alpha+16\big)u^3$$
$$+ 2\big(392\alpha^2-397\alpha-218\big)u^2 - 2(134\alpha+45)u - 18\alpha - 5.$$

Using Descartes' rule of signs, it is straightforward to show that, for any $\alpha > 0$, each of the polynomials above has precisely one positive real root. The values at the endpoints of each J_i can be determined by direct substitution. It follows that $p_2 = 0$ has one solution in each of the intervals J_1, J_2, J_3, and J_4.

Polynomial p_2: The Case $m \in (-1, 0)$ Consider the intervals $K_1 = [0, \frac{1}{2}]$, $K_2 = [\frac{1}{2}, 1]$, $K_3 = [1, \frac{m+2}{m+1}]$, and $K_4 = [\frac{m+2}{m+1}, \infty)$. We show that for each $m \in (-1, 0)$, p_2 has a root in each of these intervals.

x-interval	Möbius Transformation	
K_1	$x = \frac{1}{2}\frac{u}{u+1},$	$m = -\frac{1}{\alpha+1}$
K_2	$x = \frac{u+\frac{1}{2}}{u+1},$	$m = -\frac{1}{\alpha+1}$
K_3	$x = \frac{1+\frac{m+2}{m+1}u}{u+1},$	$m = -\frac{1}{\alpha+1}$
K_4	$x = u + \frac{m+2}{m+1},$	$m = -\frac{1}{\alpha+1}$

$$P_{K_1} = \big(\alpha^2+2\alpha\big)u^4 + 2\big(\alpha^2+2\alpha\big)u^3 - 2\big(4\alpha^2+5\alpha+2\big)u^2$$
$$- 2\big(12\alpha^2+14\alpha+4\big)u - 4\big(4\alpha^2+4\alpha+1\big),$$

$$P_{K_2} = -4u^4 + 4(\alpha-2)u^3 + 2\big(2\alpha^2+7\alpha-2\big)u^2 + 6\big(\alpha^2+2\alpha\big)u + \big(\alpha^2+2\alpha\big),$$

$$P_{K_3} = 4\big(4\alpha^4 + 12\alpha^3 + 9\alpha^2 + 2\alpha\big)u^4 + 8\big(6\alpha^4 + 17\alpha^3 + 11\alpha^2 + 2\alpha\big)u^3$$
$$+ 8\big(2\alpha^4 + 2\alpha^3 - 10\alpha^2 - 9\alpha - 2\big)u^2 - 8\big(\alpha^3 + 5\alpha + 2\big)u - 4\alpha^2,$$
$$P_{K_4} = -16\alpha^3 u^4 - 8\big(6\alpha^3 + 4\alpha^2\big)u^3 - 8\big(4\alpha^3 + 5\alpha^2 + 2\alpha\big)u^2$$
$$+ 8\big(2\alpha^3 + 5\alpha^2 + 2\alpha\big)u + 4\big(4\alpha^3 + 12\alpha^2 + 9\alpha + 2\big).$$

Using Descartes' rule of signs, it is straightforward to show that, for any $\alpha > 0$, each of the polynomials above has one positive real root. Again, it is straightforward to check the behavior at the endpoints of each interval by direct substitution. It follows that $p_2 = 0$ has one solution in each of the intervals K_1, K_2, K_3, and K_4.

6.3 The Variety of Asymmetric Configurations

In this section we study the strictly planar, asymmetric solutions to our problem and show that there are exactly eight asymmetric solutions for each $m \in (-1, 1)$. We restrict to the case $\lambda' = -1$ since $\lambda' > 0$ only leads to symmetric solutions (as shown in Sect. 7.3) We first recall some definitions and theorems from algebraic geometry (see Bochnak et al. 1998 and Cox et al. 2007 for more details).

Definition 6.3 Let A be a commutative ring.

1. An ideal I of A is said to be real if, for every sequence $a_1, \ldots, a_p$ of elements of A, we have

$$a_1^2 + \cdots + a_p^2 \in I \implies a_i \in I, \quad \text{for } i = 1, \ldots, p.$$

2. $\sqrt[R]{I}$ is the smallest real ideal of A containing I and is called the real radical of the ideal I.

Let k be a field. If $I \subset k[x_1, \ldots, x_n]$ is an ideal, we denote by $\mathbf{V}(I)$ the set

$$\mathbf{V}(I) = \big\{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\big\}.$$

$\mathbf{V}(I)$ is an affine variety. In particular if $I = \langle f_1, \ldots, f_s \rangle$, then $\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s)$. If k is the field of real numbers $\mathbb{R}$, we say that $\mathbf{V}(I)$ is a real algebraic variety. Note that while this terminology is common in algebraic geometry books, it is different from the terminology frequently used in real algebraic geometry (see Bochnak et al. 1998, for example).

Definition 6.4 Let $V \subset k^n$ be an affine variety. Then we set

$$\mathbf{I}(V) = \big\{f \in k[x_1, \ldots, x_n] : f(a_1, \ldots a_n) = 0 \text{ for all } (a_1, \ldots a_n) \in V\big\}.$$

$\mathbf{I}(V)$ is an ideal and it is called the ideal of the variety V.

We are now ready to state a version of the Real Nullstellensatz:

Theorem 6.5 (Real Nullstellensatz) *Let k be a real closed field and I an ideal of $k[x_1, \ldots, x_n]$. Then $\mathbf{I}(\mathbf{V}(I)) = \sqrt[R]{I}$.*

The dimension of an affine variety $V \subset k^n$, denoted $\dim V$, is the degree of the affine Hilbert polynomial of the corresponding ideal $\mathbf{I}(V) \subset k[x_1, \ldots, x_n]$ (see Cox et al. 2007 for more details). This degree can easily be computed using SINGULAR. In the case $k = \mathbb{R}$, one needs to know the ideal of the variety, or by the Real Nullstellensatz, the real radical of the ideal. The real radical can be computed using the `realrad.lib` (Spang 2011) library of SINGULAR (Decker et al. 2011). More details about the algorithms can be found in the paper Spang (2008). We now apply the above theory to our problem.

Theorem 6.6 *Consider the ring $\mathbb{R}[s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}, m]$. The asymmetric relative equilibria configurations form a one-dimensional real variety $V \subset \mathbb{R}^7$.*

Proof Consider the Gröbner basis G_{I_s} for the ideal I_s and saturate with respect to s_{13}, $s_{14}, s_{24}, (s_{13} - s_{24}), (s_{14} - s_{23}), (s_{13} - s_{14}), (s_{23} - s_{24}), (s_{13} - s_{23})$, and $(s_{14} - s_{24})$, as before. In addition, saturate with respect to $(ms_{12} - 2m + s_{12} - 1)$ to eliminate the case $s_{12} = (2m + 1)/(m + 1)$ for which the solutions are complex. We obtain the following polynomials:

$$f_1 = s_{13} + s_{14} + s_{23} + s_{24} - 2s_{34} - 1,$$

$$f_2 = s_{12} - s_{34} + 1,$$

$$f_3 = s_{34}m + s_{34} - m - 2,$$

$$f_4 = 2s_{24}^2 - s_{14}s_{34} - s_{23}s_{34} - 4s_{24}s_{34} + 2s_{34}^2 + 2s_{23},$$

$$f_5 = 2s_{23}s_{24} - 2s_{23} - 2s_{24} + s_{34},$$

$$f_6 = 2s_{14}s_{24} - s_{34},$$

$$f_7 = 2s_{23}^2 + s_{14}s_{34} - 3s_{23}s_{34} + 2s_{24} - s_{34},$$

$$f_8 = 2s_{14}s_{23} - s_{14}s_{34} - s_{23}s_{34} + s_{34},$$

$$f_9 = 2s_{14}^2 - 3s_{14}s_{34} + s_{23}s_{34} - 2s_{14} - 2s_{23} - 2s_{24} + 3s_{34} + 2.$$

Let $I = \langle f_1, \ldots, f_9 \rangle$. We want to find the dimension of the real variety $\mathbf{V}(I)$. First, we find $\mathbf{I}(\mathbf{V}(I))$ or $\sqrt[R]{I}$. This can be computed using the `realrad.lib` (Spang 2011) library of SINGULAR (Decker et al. 2011). However, in this case it turns out that $\sqrt[R]{I} = I$, and hence $f_1, \ldots, f_9$ are the generators of the real radical of I. The degree of the affine Hilbert polynomial of $\sqrt[R]{I} = I$ is one, and thus $\mathbf{V}(I)$ is a one-dimensional real algebraic variety. $\qquad \square$

There are some important solutions that can be determined directly from equations $f_1 = 0, \ldots, f_9 = 0$ when $m = 0$ or $m = 1$. For $m = 0$, there are four solutions given by

$$s_{34} = 2, \qquad s_{12} = 1, \qquad s_{14} = s_{24} = 1, \qquad s_{13}, s_{23} = \frac{3 \pm \sqrt{5}}{2} \qquad (24)$$

and

$$s_{34} = 2, \qquad s_{12} = 1, \qquad s_{13} = s_{23} = 1, \qquad s_{14}, s_{24} = \frac{3 \pm \sqrt{5}}{2}. \qquad (25)$$

The points in (24) correspond to configurations where vortices $1, 2,$ and 3 are collinear and vortices $1, 2,$ and 4 form an equilateral triangle. For one point, vortex 2 lies between vortices 1 and 3, while for the other point, vortex 1 lies between vortices 2 and 3. The points in (25) correspond to the same configurations, but with vortices 3 and 4 interchanged. Although these solutions are not physically relevant, they are significant geometrically since they correspond to "boundary" configurations between the concave and convex solutions.

There are two solutions for the case $m = 1$, each with multiplicity two. They are given by

$$s_{12} = s_{13} = s_{14} = \frac{1}{2}, \qquad s_{23} = s_{24} = s_{34} = \frac{3}{2} \qquad (26)$$

and

$$s_{12} = s_{23} = s_{24} = \frac{1}{2}, \qquad s_{13} = s_{14} = s_{34} = \frac{3}{2}. \qquad (27)$$

The point in (26) corresponds to a concave solution where vortices $2, 3,$ and 4 are located at the vertices of an equilateral triangle and vortex 1 is at the center. The point in (27) is the same configuration, but with vortex 2 at the center. The fact that they are double roots can be shown using Gröbner bases. These symmetric configurations have not been eliminated because in the case of equal vorticities, there is an additional symmetry that we did not exclude when saturating.

We now determine if the variety contains any singular points. First, we recall some definitions and theorems.

Definition 6.7 Let $V \subset k^n$ be an affine algebraic variety. The Zariski tangent space of V at $p = (p_1, \ldots, p_n)$, denoted by $T_p(V)$, is the variety

$$T_p(V) = \mathbf{V}\big(d_p(f) : f \in \mathbf{I}(V)\big),$$

where $d_p(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i)$.

Clearly if $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$ then $T_p(V) = \mathbf{V}(d_p(f_1), \ldots, d_p(f_s))$, and it is the translate of a linear subspace of k^n. Recall that if $V \subset k^n$ is an affine variety, then V is irreducible if and only if $\mathbf{I}(V)$ is a prime ideal. Thus we have the following definition:

Definition 6.8 Let $V \subset k^n$ be an irreducible variety. A point p in V is nonsingular (or smooth) provided that $\dim T_p(V) = \dim V$. In other words, if $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$, then p is nonsingular if and only if the rank of the Jacobian matrix $J = [\frac{\partial f_i}{\partial x_j}]$ is equal to $n - \dim(V)$. Otherwise, p is a singular point of V.

The notion of a Zariski tangent space is defined even at a singular point. However, when $V \subset \mathbb{R}^n$ is an irreducible variety and $p \in V$ is a nonsingular point, a neighborhood of p in V is a C^∞ submanifold of $\mathbb{R}^n$, and the tangent space of V at z (in

the C^∞ sense) coincides with the Zariski tangent space. If p is singular, then the dimension of $T_p(V)$ is bigger than the dimension of V.

Theorem 6.9 *For $-1 < m \le 1$, the variety of strictly planar, asymmetric relative equilibria configurations has no singular points and hence it is a smooth one-dimensional manifold.*

Proof The Jacobian matrix of $(f_1, \ldots, f_9)$ is

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & m+1 & s_{34}-1 \\
0 & 0 & -s_{34} & -s_{34}+2 & 4(s_{24}-s_{34}) & -s_{14}-s_{23}-4(s_{24}-s_{34}) & 0 \\
0 & 0 & 0 & 2s_{24}-2 & 2s_{23}-2 & 1 & 0 \\
0 & 0 & 2s_{24} & 0 & 2s_{14} & -1 & 0 \\
0 & 0 & s_{34} & 4s_{23}-3s_{34} & 2 & s_{14}-3s_{23}-1 & 0 \\
0 & 0 & 2s_{23}-s_{34} & 2s_{14}-s_{34} & 0 & -s_{14}-s_{23}+1 & 0 \\
0 & 0 & 4s_{14}-3s_{34}-2 & s_{34}-2 & -2 & -3s_{14}+s_{23}+3 & 0
\end{bmatrix}.
$$

Using Gaussian elimination we obtain the matrix

$$
\left[
\begin{array}{cccc:ccc}
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 1 & 1 & -2 & 0 \\
0 & 0 & -s_{34} & -s_{34}+2 & 4(s_{24}-s_{34}) & -s_{14}-s_{23}-4(s_{24}-s_{34}) & 0 \\
0 & 0 & 0 & 2(s_{24}-1) & 2(s_{23}-1) & 1 & 0 \\
\hdashline
 & & & & q_1 & q_2 & 0 \\
 & & & & 0 & m+1 & s_{34}-1 \\
 & & \mathbf{0} & & 0 & 0 & q_3 \\
 & & & & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0
\end{array}
\right],
$$

where q_1, q_2, q_3 are rational functions in the s_{ij}, m variables. Note that if $m \ne 0$, then $s_{24} \ne 1 = \sqrt{-\lambda'}$, by Propositions 2.2 and 2.4. Using a Gröbner basis with the ordering $m > s_{34} > s_{12} > s_{13} > s_{24} > s_{23} > s_{14}$, q_1 simplifies to

$$
\frac{2(4s_{14}s_{23}-1-s_{14}-s_{23})}{s_{14}(s_{14}-s_{23})} \tag{28}
$$

on the variety, while q_3 simplifies to 0. Due to the symmetry lemma, the denominator of (28) cannot vanish since we are excluding symmetric solutions. Adding the numerator of (28) to $f_1, \ldots, f_9$ and computing a Gröbner bases with an ordering that eliminates all the s_{ij} variables produces a quadratic polynomial in m with roots at $m = 1$ or $m = -5/3$. It follows that, for the points on the variety with $m \in (-1, 0) \cup (0, 1)$, the rank of the Jacobian matrix is 6. The specific points for $m = 0$ and $m = 1$ given in Eq. (24) through (27) can be checked individually to verify that the rank of the Jacobian matrix is 6 for these cases as well. Since $\dim(V) = 1$, by Definition 6.8 the variety for $-1 < m \le 1$ has no singular points. It follows that it is a smooth manifold. $\square$

Fig. 5 Projection of part of the manifold of asymmetric configurations onto the ms_{23}-plane. As $m \to -1$, $s_{23} \to \infty$ along two branches while $s_{23} \to \frac{1}{2}$ along the other two branches
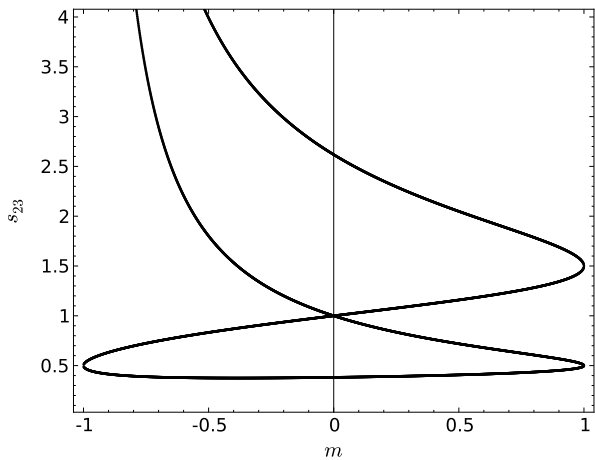
A projection of the smooth manifold of asymmetric relative equilibria onto the ms_{23}-plane is shown in Fig. 5. Note the crossing at $(m = 0, s_{23} = 1)$ corresponding to the two "boundary" points given by (25). Also note the double roots visible at $(m = 1, s_{23} = 1/2)$ and $(m = 1, s_{23} = 3/2)$.

Theorem 6.10 *For each value of m in $(-1, 1)$, there are exactly four (eight counting reflected solutions) strictly planar, asymmetric relative equilibria configurations. They are convex if $m < 0$ and concave if $m > 0$.*

Proof The proof consists of showing that the system of equations $f_1 = 0, \ldots, f_9 = 0$ has four solutions for each value of m in $(-1, 1)$. The convexity of the configuration then follows from Theorem 6.2. The case $m = 0$ has already been proven directly. We use an algorithm for the triangular decomposition of semi-algebraic systems. Such an algorithm, given a system of equations and inequalities S, computes simpler systems $S_1, \ldots, S_k$ such that a point is a solution of the original system S if and only if it is a solution of one of the systems $S_1, \ldots, S_k$. Each of these systems has a triangular shape and remarkable properties: for this reason it is called a regular semi-algebraic system and the set of the $S_1, \ldots, S_k$ is called a full triangular decomposition of S. See Chen et al. (2010) and references therein for some background. The algorithm is detailed in Chen et al. (2010) and it is available in Maple15$^{\text{TM}}$ via the `RealTriangularize` command of the `RegularChains` package.

For $m \in (-1, 0) \cup (0, 1)$, using the `RealTriangularize` command and imposing the constraint that all $s_{ij} > 0$, the triangular decomposition of the system $f_1 = 0, \ldots, f_9 = 0$ is

$$s_{12} - s_{34} + 1 = 0,$$

$$s_{13} + s_{14} + s_{23} + s_{24} - 2s_{34} - 1 = 0,$$

$$s_{34}s_{14} + (s_{34} - 2)s_{23} - 2s_{24}^2 + 4s_{24}s_{34} - 2s_{34}^2 = 0,$$

$$(2s_{24} - 2)s_{23} - 2s_{24} + s_{34} = 0,$$

$$4s_{24}^4 + (-8s_{34} - 4)s_{24}^3 + \left(4s_{34}^2 + 6s_{34} + 4\right)s_{24}^2 + \left(-4s_{34}^2 - 2s_{34}\right)s_{24} + s_{34}^2 = 0,$$

$$(m+1)s_{34} - m - 2 = 0.$$

Solving the last equation gives $s_{34} = \frac{m+2}{m+1}$. Substituting this into the preceding equation and clearing denominators yields $p_2(s_{24})$. From Lemma 6.1 we know that p_2 has four distinct positive real roots for $m \in (-1, 0) \cup (0, 1)$. If we fix one such solution for s_{24} and substitute into the first four equations of the system, the resulting system can be written as $Ax = b$, where $x = (s_{12}, s_{13}, s_{14}, s_{23})$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & s_{34} & (s_{34} - 2) \\ 0 & 0 & 0 & 2(s_{24} - 1) \end{bmatrix}$$

and $b = [-1 + s_{34}, -s_{24} + 2s_{34} + 1, 2s_{24}^2 - 4s_{24}s_{34} + 2s_{34}^2, 2s_{24} - s_{34}]^T$. Since the matrix A is invertible, $Ax = b$ has exactly one solution for every vector b. This proves the theorem since it implies that the system $f_1 = 0, \dots f_9 = 0$ has four real positive solutions. $\qquad\square$

Remark Since the variety of asymmetric relative equilibria contains no singular points and the number of solutions is constant for $-1 < m < 1$, it follows that the convex configurations for $m < 0$ smoothly transform into concave configurations as m increases through 0. The solutions at $m = 0$ consisting of three collinear vortices supports this observation. An animation of this interesting geometric bifurcation was created in Maple15™ and can be viewed at http://mathcs.holycross.edu/~groberts/Research/vort-movies.html.

7 Symmetric Relative Equilibria

In this section we investigate all of the strictly planar, symmetric relative equilibria in the four-vortex problem with $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. The four possible configurations are an isosceles trapezoid, a concave kite, a convex kite, or a rhombus. When $m = 0$, it is possible to have a solution where precisely three of the four vortices are collinear. We separate the kite configurations into two cases based on the sign of λ'.

7.1 The Isosceles Trapezoid Family

Without loss of generality, suppose that the vortices are ordered sequentially around an isosceles trapezoid, so that the lengths of the diagonals are given by $r_{13} = r_{24}$ and the congruent legs have length $r_{14} = r_{23}$. Due to the symmetry of the configuration, the four vortices lie on a common circle, that is, the isosceles trapezoid is a cyclic quadrilateral. By Ptolemy's theorem, we have

$$r_{13}^2 = r_{24}^2 = r_{12}r_{34} + r_{14}^2. \tag{29}$$

A choice of six mutual distances satisfying $r_{13} = r_{24}$, $r_{14} = r_{23}$, and Eq. (29) will also satisfy $e_{CM} = 0$.

In contrast to the Newtonian four-body problem, the isosceles trapezoid family of four-vortex relative equilibria can be solved completely (by hand) in terms of the vortex strength m (compare with Cors and Roberts 2012). Let $x = r_{34}/r_{12}$ and $y = r_{14}/r_{12}$. Rather than assuming a specific scaling on the distances (e.g., $\lambda' = -1$), we will solve for x and y in terms of m. Using the symmetries of the configuration and Eq. (29), Eq. (10) reduces to

$$2(y^2 - x^2)(y^2 - 1) + x(2y^2 - x^2 - 1) = 0. \tag{30}$$

This relation between x and y is both necessary and sufficient for the trapezoid to be a relative equilibrium.

Focusing on the first and last columns of Eq. (9), the first and last equations are easily satisfied due to symmetry. The middle four equations in (9) are equivalent given that (30) holds. They determine a formula for m given by

$$m = \frac{x(1 - y^2)}{y^2 - x^2}, \tag{31}$$

where we have used $|A_i| = r_{jk}r_{jl}r_{kl}/(4r_c)$ for the area of a triangle circumscribed in a circle of radius r_c. Solving Eq. (31) for y^2 and substituting into Eq. (30) leads to the equation

$$x(1 - x)(1 + x)\big(x^2(2m + 1) - m(m + 2)\big) = 0. \tag{32}$$

Since x represents the ratio of two distances, there are only two possibilities. If $x = 1$, then Eq. (30) quickly gives $y = 1$ and the configuration is a square. The second equation in (9) then gives $m = 1$ and all vortices must have the same strength. The other possibility is that $x^2 = m(m + 2)/(2m + 1)$. Substituting this value into Eq. (31) and solving for y^2 yields

$$y^2 = \frac{1}{2}\left(m + 2 - \sqrt{\frac{m(m + 2)}{2m + 1}}\right).$$

Note that the necessary condition $x^2 > 0$ is satisfied only for $m > 0$ and $-2 < m < -1/2$. In order for the expression for y^2 to be real, we must be in one of these two regimes. However, while $y^2 > 0$ holds for $m > 0$, it is not satisfied on $-2 < m < -1/2$. Therefore, there does *not* exist an isosceles trapezoid solution for the case $m < 0$.

Theorem 7.1 *There exists a one-parameter family of isosceles trapezoid relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. The vortices 1 and 2 lie on one base of the trapezoid, while 3 and 4 lie on the other. Let $\alpha = m(m + 2)/(2m + 1)$. If $r_{13} = r_{24}$ are the lengths of the two congruent diagonals, then the mutual distances are described by*

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \alpha, \qquad \left(\frac{r_{14}}{r_{12}}\right)^2 = \frac{1}{2}(m + 2 - \sqrt{\alpha}) \quad and$$

$$\left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{2}(m + 2 + \sqrt{\alpha}).$$

This family exists if and only if $m > 0$. The case $m = 1$ reduces to the square. For $m \neq 1$, the larger pair of vortices lie on the longer base.

Proof The formula for r_{13}^2/r_{12}^2 comes from Eq. (29). Substituting $m = 1$ into the formulas above quickly gives $r_{12} = r_{14} = r_{34} = r_{23}$ and $r_{13} = r_{24} = \sqrt{2}r_{12}$, a square. The only other item that remains to be shown is the fact about the larger vortices lying on the longer base. This follows since $\alpha > 1$ if and only if $m > 1$.   □

Remark

1. If $0 < m < 1$, r_{12} is the longest base length and the formulas in the theorem give $r_{34} < r_{14} < r_{12} < r_{13}$. On the other hand, if $m > 1$, then r_{34} is the longest base length and we deduce $r_{12} < r_{14} < r_{34} < r_{13}$. Both cases agree with the conclusion of statement 1 in Proposition 2.4.

2. It can be shown rigorously that the isosceles trapezoid relative equilibria are linearly stable for all $m > 0$ (Roberts 2013). In fact, the periodic solutions are actually nonlinearly stable as well (see Theorem 3.8 in Roberts 2013). In other words, solutions starting sufficiently close to an isosceles trapezoid central configuration will remain close to the corresponding relative equilibrium (as a set) for all forward and backward time.

3. The case $m > 1$ is identical to the case $0 < m < 1$ through a rescaling of the vortex strengths and interchanging bodies 1 and 3, and bodies 2 and 4. Specifically, replacing m by $1/m$ and interchanging distances r_{12} and r_{34} leaves the three equations for the ratios of mutual distances unchanged. The vortex strengths map to $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = 1/m < 1$ under this transformation.

4. As $m \to 0$, $\alpha \to 0$ and consequently $r_{34} \to 0$. The limiting configuration is an equilateral triangle with vortices 3 and 4 colliding. As m increases, so does α, and the smaller base length r_{34} approaches the larger one r_{12} as $m \to 1$. The ratio of the diagonal to the larger base also increases monotonically with m. However, the ratio of the leg to the larger base begins at 1 ($m = 0$) and decreases to a minimum value of approximately 0.904781 before increasing back to 1 at the square configuration. This minimum value occurs at $m \approx 0.234658$, which is the only positive root of the quartic $8m^4 + 19m^3 + 9m^2 + m - 1$. Just as with the Newtonian case (see Cors and Roberts 2012), the range of y is surprisingly small, confined to the interval [0.904781, 1].

7.2 Kite Configurations: $\lambda' < 0$

In this section, we consider kite central configurations when $\lambda' < 0$. Such configurations contain two vortices that are symmetrically located with respect to an axis of symmetry while the remaining two vortices lie on the axis of symmetry. The configuration formed by the vortices can either be concave or convex. In this section,

we focus on configurations with only one line of symmetry. Those configurations with two lines of symmetry, i.e., the rhombus configurations, are studied in detail in Sect. 7.4.

Our goal is to count the number of kite configurations as m varies and describe the type of possible configurations. Recall that $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \tilde{e}_{\mathrm{CM}}$ and $\widetilde{\mathcal{H}}$ represent the Albouy–Chenciner equations, the unsymmetrized Albouy–Chenciner equations, the Cayley–Menger determinant and the Dziobek equations, respectively, in terms of the variables $s_{ij} = r_{ij}^2$, with $\lambda' = -1$. Denote the complete system of these equations as $\widetilde{\mathcal{E}}$. Our goal is to prove the following theorem.

Theorem 7.2 *For each value of $m \in (-\infty, -2) \cup (-\frac{1}{2}, 0) \cup (0, \infty)$, system $\widetilde{\mathcal{E}}$ has exactly four (real, positive) solutions that correspond to symmetric kite configurations. If $m \in (0, \infty)$, such configurations are concave. If $m \in (-\frac{1}{2}, 0)$, then there are two convex configurations with Γ_3 and Γ_4 on the axis of symmetry, and two concave ones with Γ_1 and Γ_2 on the axis of symmetry. If $m \in (-\infty, -2)$, the situation is reversed and there are two concave configurations with Γ_3 and Γ_4 on the axis of symmetry, and two convex ones with Γ_1 and Γ_2 on the axis of symmetry. For $m = 1$, the system has exactly four (real positive) solutions that correspond to an equilateral triangle with a vortex at the center. There are no other strictly kite configurations with $\lambda' < 0$ for other values of m.*

Remark The exact count on the number of kite solutions given in Table 1 is precisely twice the values expressed in the theorem. This is due to the fact that each solution found can be reflected about the axis of symmetry, a transformation that does not change the mutual distances r_{ij}, but does alter the positions x_i.

We have two possibilities: either vortices 3 and 4 lie on the axis of symmetry or vortices 1 and 2 do. It is convenient to study the two cases separately, since the conditions imposed by the symmetry are different. The first case will be explored in Lemma 7.3, and the second in Lemma 7.4. The proof of Theorem 7.2 follows immediately from the two lemmas.

Let us consider the first case. If vortices 3 and 4 lie on the axis of symmetry, then $s_{23} = s_{13}$ and $s_{24} = s_{14}$. We compute a Gröbner basis for the ideal $\langle \bar{\mathcal{F}}, \bar{\mathcal{G}}, \bar{e}_{\mathrm{CM}}, \bar{H} \rangle$, where $\bar{\mathcal{F}}, \bar{\mathcal{G}}, \bar{e}_{\mathrm{CM}}, \bar{\mathcal{H}}$ are obtained from $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \tilde{e}_{\mathrm{CM}}$ and $\widetilde{\mathcal{H}}$, respectively, by imposing the conditions $s_{23} = s_{13}$ and $s_{24} = s_{14}$. Then we saturate with respect to $s_{13}-s_{14}$, in order to exclude the rhombus configurations. We also saturate with respect to $s_{12}, s_{13}, s_{14}, s_{34}$, and m. This computation yields the following ten polynomials:

$$g_1 = ms_{34} + s_{13} + s_{14} + s_{34} - m - 2,$$

$$g_2 = ms_{13} + ms_{14} + s_{12} + s_{13} + s_{14} + s_{34} - 3m - 3,$$

$$g_3 = 2s_{12}s_{34} - 2s_{13}s_{34} - 2s_{14}s_{34} + 2s_{34}^2 - s_{12} + 2s_{13} + 2s_{14} + s_{34} - 3,$$

$$g_4 = 2s_{13}s_{14} - s_{13} - s_{14} + s_{34},$$

$$g_5 = 2s_{13}^2 + 2s_{14}^2 - 2s_{13}s_{34} - 2s_{14}s_{34} + s_{12} - 4s_{13} - 4s_{14} + s_{34} + 3,$$

$$g_6 = 2s_{12}s_{13} + 2s_{12}s_{14} - ms_{12} - 2s_{12} - s_{13} - s_{14} + s_{34},$$

$$g_7 = m^2 s_{12} + 2s_{12}^2 + 2s_{13}s_{34} + 2s_{14}s_{34} - 2s_{34}^2 - 3ms_{12} - 4s_{12}$$
$$\quad - s_{13} - s_{14} - 2s_{34} + 2m + 4,$$
$$g_8 = 2ms_{14}^2 + 2s_{12}s_{14} + 2s_{14}^2 + 2s_{14}s_{34} - 6ms_{14} - s_{12} + s_{13} - 5s_{14} - s_{34}$$
$$\quad + 2m + 1,$$
$$g_9 = 4s_{14}^3 - 4s_{14}^2 s_{34} + 2s_{12}s_{14} - 10s_{14}^2 - 2s_{13}s_{34} + 2s_{14}s_{34} + 2s_{34}^2 - s_{12}$$
$$\quad + s_{13} + 7s_{14} + 2s_{34} - 3,$$
$$g_{10} = 4s_{12}s_{14}^2 - 2ms_{12}s_{14} - 4s_{12}s_{14} - 2s_{14}^2 - 2s_{13}s_{34} + 2s_{34}^2 + ms_{12} + s_{12}$$
$$\quad + 2s_{13} + 2s_{14} + s_{34} - 3.$$

Let us denote the corresponding system of equations $g_1 = 0, \ldots, g_{10} = 0$ as G. Then the positive solutions of G give the symmetric kite configurations described above. An exact count of these solutions is described in the following lemma.

Lemma 7.3 *For each value of $m \in (0, 1)$, system G has exactly four real positive solutions corresponding to concave configurations. For each value of $m \in (-\infty, -2) \cup (-\frac{1}{2}, 0)$, system G has exactly two real positive solutions. Such solutions correspond to convex configurations if $m \in (-\frac{1}{2}, 0)$ and to concave configurations if $m \in (-\infty, -2)$. For $m = 1$, system G has exactly two real positive solutions corresponding to an equilateral triangle with either vortex 3 or 4 at the center. There are no positive real solutions for other values of m.*

Proof Computing a full triangular decomposition of system G with the positivity conditions $s_{12} > 0$, $s_{13} > 0$, $s_{14} > 0$, and $s_{34} > 0$, we obtain five simpler systems. If $m = 1$, we have the following two systems:

$$T_1 = \begin{cases} 2s_{12} - 3 = 0, \\ 2s_{13} - 3 = 0, \\ 2s_{14} - 1 = 0, \\ 2s_{34} - 1 = 0, \end{cases} \qquad T_2 = \begin{cases} 2s_{12} - 3 = 0, \\ 2s_{13} - 1 = 0, \\ 2s_{14} - 3 = 0, \\ 2s_{34} - 1 = 0. \end{cases}$$

If $m = 0$, we have the following two systems:

$$T_3 = \begin{cases} s_{12} - 1 = 0, \\ s_{13} - 1 = 0, \\ 4s_{14} - 1 = 0, \\ 4s_{34} - 3 = 0, \end{cases} \qquad T_4 = \begin{cases} s_{12} - 1 = 0, \\ 4s_{13} - 1 = 0, \\ s_{14} - 1 = 0, \\ 4s_{34} - 3 = 0. \end{cases} \qquad (33)$$

If $m \in (-\infty, -2) \cup (-1/2, 0) \cup (0, 1)$, we obtain the system

$$T_5 = \begin{cases} s_{12} + (-m^2 - 2m)s_{34} - 1 + m^2 = 0, \\ s_{13} + s_{14} + (m+1)s_{34} - 2 - m = 0, \\ 2s_{14}^2 + ((2m+2)s_{34} - 4 - 2m)s_{14} + (-2 - m)s_{34} + m + 2 = 0, \\ (4 + 6m + 2m^2)s_{34}^2 + (-3 - 3m^2 - 6m)s_{34} + 2m + m^2 = 0. \end{cases}$$

The case $m = 1$ is straightforward, since T_1 and T_2 each have a unique solution corresponding to an equilateral triangle with either vortex 3 or 4 at the center. The case $m = 0$, although physically meaningless, has two important solutions that divide the concave configurations from the convex ones. In the first solution, vortex 4 is collinear with vortices 1 and 2, lying equidistant from each of them. Vortex 3 forms an equilateral triangle with vortices 1 and 2. The second solution for $m = 0$ is the same configuration with vortices 3 and 4 interchanged. It remains to study the solutions of system T_5.

- Solutions of T_5 with $s_{34} > 0$ and $s_{14} > 0$

The first two equations of T_5 are linear in s_{12} and s_{13}, so that to each solution (s_{14}, s_{34}) of the last two equations correspond one solution $(s_{12}, s_{13}, s_{14}, s_{34})$ of T_5, provided that the determinant of the matrix of the coefficients of the linear system is not zero. Hence, we focus our attention on the last two equations of T_5. Let $A = 2(2 + 3m + m^2)$, $B = -3(m+1)^2$, and $C = m(2 + m)$. Then the last equation of T_5 can be written as

$$q_1(s_{34}) = As_{34}^2 + Bs_{34} + C = 0$$

and can be viewed as a parametric equation of a parabola. Clearly $B < 0$ for all $m \neq -1$, and $A > 0$ for $m \in (-\infty, -2) \cup (-1, \infty)$. Thus $A > 0$, $B < 0$ on the domain where T_5 is valid. The vertex of the parabola has coordinates

$$\left(-\frac{B}{2A}, q_1\left(-\frac{B}{2A}\right) \right) = \left(\frac{3(m+1)^2}{2(2 + 3m + m^2)}, -\frac{1}{8}\frac{(m-1)(m^2 - 4m - 9)}{m+2} \right).$$

It is easy to see that $-\frac{B}{2A} > 0$ and $q_1(-\frac{B}{2A}) < 0$ in the interval of interest, and hence the equation has two real solutions for each value of $m \in (-\infty, -2) \cup (-1/2, 0) \cup (0, 1)$. Since $C > 0$ for $m \in (-\infty, -2) \cup (0, \infty)$ and $C < 0$ for $m \in (-2, 0)$, it follows that the equation has two positive solutions for $m \in (-\infty, -2) \cup (0, 1)$ and only one positive solution for $m \in (-1/2, 0)$.

Let us denote as β_1 and β_2 (with $\beta_1 < \beta_2$) the (real) solutions of $q_1(s_{34}) = 0$. Then through a standard analysis it is possible to find bounds for β_1 and β_2. Such bounds are summarized in the table below.

Solutions	$m \in (-\infty, -2)$	$m \in (-\frac{1}{2}, 0)$	$m \in (0, 1)$
β_1	$0 < \beta_1 < \frac{1}{2}$	$\beta_1 < 0$	$0 < \beta_1 < \frac{1}{2}$
β_2	$\beta_2 > 1$	$0 < \beta_2 < 1$	$0 < \beta_2 < 1$

Now consider the third equation of system T_5. Let $A' = 2$, $B' = (2(m+1)s_{34} - 4 - 2m)$, and $C' = (m+2)(1 - s_{34})$. Then this equation takes the form

$$q_2(s_{14}) = A's_{14}^2 + B's_{14} + C' = 0.$$

Since this is a quadratic equation in s_{14}, there are going to be two solutions for each s_{34} and m. We will denote the (real) solutions corresponding to $s_{34} = \beta_1$ as β_1^1 and β_1^2 (with $\beta_1^1 < \beta_1^2$), and the ones corresponding to $s_{34} = \beta_2$ as β_2^1 and β_2^2 (with $\beta_2^1 < \beta_2^2$). The vertex of this parabola has coordinates

$$\left(-\frac{B'}{2A'}, q_2\left(-\frac{B'}{2A'} \right) \right)$$
$$= \left(\frac{(m+1)(1 - s_{34}) + 1}{2}, -\frac{1}{2}(m+1)^2 s_{34}^2 + m(m+2)s_{34} - \frac{m}{2}(m+2) \right).$$
(34)

We can then view the ordinate of the vertex, denoted as $r(s_{34})$, as a parabola with coefficients in m:

$$r(s_{34}) = -\frac{1}{2}(m+1)^2 s_{34}^2 + m(m+2)s_{34} - \frac{m}{2}(m+2) = A''s_{34}^2 + B''s_{34} + C'' = 0.$$

The coordinates of the vertex of $r(s_{34})$ are

$$\left(-\frac{B''}{2A''}, r\left(-\frac{B''}{2A''} \right) \right) = \left(\frac{m(m+2)}{(m+1)^2}, -\frac{1}{2}\frac{m(m+2)}{(m+1)^2} \right).$$

Case I: $m \in (0, 1)$

Since $-\frac{B''}{2A''} > 0$, $r(-\frac{B''}{2A''}) < 0$, and $A'' < 0$ for $m \in (0, 1)$, it follows that $q_2(-\frac{B'}{2A'}) < 0$ for $m \in (0, 1)$. Consequently, since $C' > 0$ in the same interval, $q_2 = 0$ has two positive solutions corresponding to each solution of $q_1 = 0$ for $m \in (0, 1)$. In other words, there are four (positive) possible values for s_{14}, namely $\beta_1^1 > 0$, $\beta_1^2 > 0$, $\beta_2^1 > 0$, and $\beta_2^2 > 0$.

Case II: $m \in (-\frac{1}{2}, 0)$

For $m \in (-\frac{1}{2}, 0)$, we have $A' > 0$ and $C' > 0$. This implies that corresponding to $s_{34} = \beta_2$, there are either two real positive solutions or no real solutions. We want to show that there are two positive real solutions. Clearly $-\frac{B''}{2A''} < 0$, $r(-\frac{B''}{2A''}) > 0$, $A'' < 0$, and $C'' > 0$ for $m \in (-\frac{1}{2}, 0)$, so that $r = 0$ has one positive and one negative root. If we show that the positive root is less than $s_{34} = \beta_2$, then it follows that $q_2(-\frac{B'}{2A'}) < 0$, and consequently $q_2 = 0$ has two positive solutions when $s_{34} = \beta_2$ (i.e., $\beta_2^1 > 0$ and $\beta_2^2 > 0$).

Subtracting β_2 from the positive root of $r = 0$ yields

$$\Delta = \frac{-(-m^3 - 7m^2 - 4xm + ym - 7m - 8x + y + 3)}{4(m+2)(m+1)^2},$$

where $x = \sqrt{-m(m+2)}$, and $y = \sqrt{(m-1)(m+1)(m^2-4m-9)}$. Numerically it is easy to see that $\Delta < 0$ for $m \in (-\frac{1}{2}, 0)$. However, we proceed more rigorously. Taking the numerator of the expression above and squaring the expressions for x and y yields three polynomials. Computing a lex Gröbner basis for these three polynomials, we obtain

$$m(5m^3 + 7m^2 + 3m + 9)(m+2)^2(m+1)^2$$

as one of the basis elements. It is trivial to see that this polynomial has no roots in $[-\frac{1}{2}, 0]$, that $\Delta(0) < 0$, and that $\Delta(-\frac{1}{2}) < 0$. Since Δ is continuous on $[-1/2, 0]$, we deduce that $\Delta < 0$ for $m \in (-\frac{1}{2}, 0)$, as desired.

Case III: $m \in (-\infty, -2)$

If $s_{34} = \beta_1$ and $m \in (-\infty, -2)$, then $C' < 0$. Since $A' > 0$, it follows that $q_2 = 0$ has one positive and one negative real solutions corresponding to β_1. If $s_{34} = \beta_2$ and $m \in (-\infty, -2)$, then $C' > 0$. Since $r(-\frac{B''}{2A''}) < 0$ and $A'' < 0$ it follows that $q_2(-\frac{B'}{2A'}) < 0$. This together with the fact that $-\frac{B'}{2A'} > 0$, $A' > 0$ and $C' > 0$ implies that, corresponding to β_2, $q_2 = 0$ has two real positive solutions, (i.e., $\beta_2^1 > 0$, and $\beta_2^2 > 0$).

- Solutions of T_5 with $s_{34} > 0$, $s_{14} > 0$, and $s_{13} > 0$

We now study which solutions among the ones found above have $s_{13} > 0$. The resultant of the second and third equations of T_5 with respect to s_{12} is

$$R = 2s_{13}^2 + ((2m+2)s_{34} - 2m - 4)s_{13} + (-2 - m)s_{34} + m + 2.$$

If s_{13} is replaced with s_{14}, this polynomial transforms into the one that appears in the third equation of T_5. Hence our discussion regarding s_{14} applies to s_{13}. Note that since we saturated with respect to $s_{13} - s_{14}$, every solution must have $s_{13} \neq s_{14}$. Since $R = 0$ has two solutions for each m and s_{34}, when s_{14} is one such solution, s_{13} must be the other.

- Solutions of T_5 with $s_{34} > 0$, $s_{14} > 0$, $s_{13} > 0$, and $s_{12} > 0$

We now study which solutions among the ones found above have $s_{12} > 0$. Consider the first equation of T_5 and rewrite it in the form

$$s_{12} = (m^2 + 2m)s_{34} + 1 - m^2. \qquad (35)$$

This equation has a unique solution for each value of s_{34} and m. Solutions corresponding to $s_{34} = \beta_i$ will be denoted α_i. We can view this expression as defining a function of the two variables m and s_{34}. The sign of this function, i.e., the sign of s_{12}, is summarized in Fig. 6. As mentioned earlier, the last equation of T_5 gives $s_{34} = \beta_1$ or $s_{34} = \beta_2$, and when $\beta_i > 0$, we have $0 < s_{34} < 1$ for $m \in (-\frac{1}{2}, 0) \cup (0, 1)$. Hence $s_{12} > 0$ for any positive solution of $q_1(s_{34}) = 0$.

The situation is a bit more delicate when $m \in (-\infty, -2)$. In this case, $\beta_2 > 1$, and thus we cannot arrive at a definitive conclusion just looking at Fig. 6. However, using standard methods, one can prove that the function of m, defined by

$$d(m) = \beta_2(m) - \frac{m^2 - 1}{m(m+2)},$$
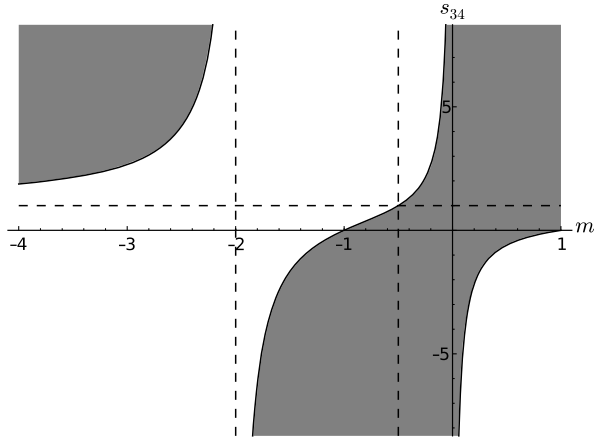
is always positive (where the graph of $\hat{f}(m) = \frac{m^2-1}{m(m+2)}$ is the boundary of the gray region in Fig. 6 in the interval $(-\infty, -2)$). Thus, β_2 lies in the region where $s_{12} > 0$ when $m \in (-\infty, -2)$. Consequently, there are always positive solutions of T_5 corresponding to β_2. On the other hand, $0 < \beta_1 < \frac{1}{2}$ for $m \in (-\infty, -2)$, and thus there are no positive solutions of T_5 corresponding to β_1 in this interval.

In summary, we have shown that the solutions $(s_{12}, s_{13}, s_{14}, s_{34})^{\mathrm{T}}$ with all components positive have the form

$$\begin{bmatrix} \alpha_1 \\ \beta_1^2 \\ \beta_1^1 \\ \beta_1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_1 \\ \beta_1^1 \\ \beta_1^2 \\ \beta_1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_2 \\ \beta_2^2 \\ \beta_2^1 \\ \beta_2 \end{bmatrix}, \quad \begin{bmatrix} \alpha_2 \\ \beta_2^1 \\ \beta_2^2 \\ \beta_2 \end{bmatrix},$$

for $m \in (0, 1)$, while they have the form

$$\begin{bmatrix} \alpha_2 \\ \beta_2^2 \\ \beta_2^1 \\ \beta_2 \end{bmatrix}, \quad \begin{bmatrix} \alpha_2 \\ \beta_2^1 \\ \beta_2^2 \\ \beta_2 \end{bmatrix}$$

for $m \in (-\infty, -2) \cup (-\frac{1}{2}, 0)$.

We are left with deciding which solutions are convex and which are concave. For $m \in (0, \infty)$, it follows from the main result in Albouy et al. (2008) that, if the kite is convex, it must be a rhombus. Since the rhombus case was excluded, these solutions correspond to concave configurations. For $m \in (-\frac{1}{2}, 0)$, we have $s_{34} = \beta_2 < 1$. However, according to Proposition 2.2, since $\lambda' = -1$, the interior side connecting equal vorticities in a concave configuration is greater than 1. Hence, the configurations are, in this case, convex. For $m \in (-\infty, -2)$ we have $s_{34} = \beta_2 > 1$. Furthermore, substituting $s_{34} = \beta_2(m)$ into Eq. (35), we obtain an expression that is a function of m alone. Examining this function, it is easy to show that $s_{12} = \alpha_2(m) < 1$ for all $m \in (-\infty, 0)$. Thus, from Propositions 2.2 and 2.4, it follows that the solu-

tions correspond, in this case, to concave configurations. This completes the proof of Lemma 7.3. $\qquad\square$

We now consider the other case, where vortices 1 and 2 lie on the axis of symmetry. We compute a Gröbner basis for the ideal $\langle \bar{\mathcal{F}}', \bar{\mathcal{G}}', \bar{e}'_{\mathrm{CM}}, \bar{\mathcal{H}}' \rangle$, where $\bar{\mathcal{F}}', \bar{\mathcal{G}}', \bar{e}'_{\mathrm{CM}}, \bar{\mathcal{H}}'$ are obtained from $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}, \widetilde{e}_{\mathrm{CM}}$ and $\widetilde{\mathcal{H}}$, respectively, by imposing the conditions $s_{14} = s_{13}$ and $s_{24} = s_{23}$. We saturate with respect to $s_{13} - s_{23}$ in order to exclude the rhombus configurations. We also saturate with respect to $s_{12}, s_{13}, s_{23}, s_{34}$, and m. This computation yields a polynomial system that we will call G_1. Then we can prove the following lemma.

Lemma 7.4 *For each value of $m \in (1, \infty)$, system G_1 has exactly four real positive solutions corresponding to concave configurations. For each value of $m \in (-\infty, -2) \cup (-\frac{1}{2}, 0)$, system G_1 has exactly two real positive solutions. Such solutions correspond to concave configurations if $m \in (-\frac{1}{2}, 0)$, and to convex configurations if $m \in (-\infty, -2)$. For $m = 1$, system G_1 has exactly two real positive solutions corresponding to an equilateral triangle with either vortex 1 or 2 at the center. There are no positive real solutions for other values of m.*

Proof Computing a full triangular decomposition of G_1, with the positivity conditions $s_{12} > 0$, $s_{13} > 0$, $s_{23} > 0$, and $s_{34} > 0$, we decompose G_1 into three simpler systems S_1, S_2, and S_3. For $m = 1$, we have two systems, S_1 and S_2. They are linear, and each of them has a unique solution corresponding to an equilateral triangle configuration with vortex 1 or 2 at the center. For $m \in (-\infty, -2) \cup (-\frac{1}{2}, 0) \cup (1, \infty)$, we obtain a third system S_3. This last system, possibly after computing a lexicographic Gröbner basis, can be reduced to T_5 by performing the transformation $(m, s_{12}, s_{13}, s_{23}, s_{34}) \mapsto (\frac{1}{m}, s_{34}, s_{14}, s_{13}, s_{12})$ and clearing the denominators. This transforms each of the four polynomial equations in S_3 to the equations of T_5. Consequently, the proof follows from Lemma 7.3. $\qquad\square$

Remark

1. The proofs of Lemmas 7.3 and 7.4 yield some interesting information on the type of kite central configurations. For example, when $m \in (0, 1)$, there are solutions only when the smaller strength vortices (3 and 4) are on the line of symmetry. In this case, there are two geometrically distinct concave configurations for each choice of the central vortex. As $m \to 0^+$, one configuration has $r_{34} \to 0$ (vortices 3 and 4 colliding), while the shape of the exterior triangle becomes equilateral. The other configuration is limiting on the solutions of the two systems in (33), where three of the vortices are collinear. Although the former Kite$_{34}$ configuration disappears when $m < 0$, the latter configuration becomes convex. In fact, applying the implicit function theorem to system T_5 at $m = 0$ and at either solution of (33), we can prove rigorously that the latter concave solutions transform smoothly into the convex solutions as m decreases through 0. The convex Kite$_{34}$ configurations that exist for $-1/2 < m < 0$ have three vortices very close to being collinear, and as $m \to -1/2^+$, $r_{34} \to \infty$. Animations of the Kite$_{34}$ configurations are available at http://mathcs.holycross.edu/~groberts/Research/vort-movies.html.

2. As described in the introduction, the equilateral triangle with a central vortex bifurcates into both concave kite configurations and asymmetric concave configurations as m decreases through $m = 1$. The fact that this particular solution shows up in both Sects. 6.3 and 7.2 is hardly surprising. In fact, when solving the Albouy–Chenciner equations for $m = 1$ without any restrictions on the variables, this solution occurs four times with a multiplicity of four. The Hessian matrix $D^2(H + \lambda I)$ evaluated at this solution has a null space of dimension 2 (excluding the "trivial" vector in the direction of rotation). One can check that there are vectors in the null space corresponding to both a symmetric and asymmetric perturbation of the solution. In fact, the symmetric perturbation gives rise to the concave kite configurations described above. The central configuration consisting of four equal vortices on an equilateral triangle with a vortex at the center is a highly degenerate solution.

7.3 Kite Configurations: $\lambda' > 0$

In Sect. 2.1 we observed that, if the vorticities have different signs, we could have central configurations with $\lambda' > 0$. In this case, taking $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m < 0$, the configuration is necessarily convex with vortices 1 and 2 on one diagonal and vortices 3 and 4 on the other. In this section, we first show that these convex solutions must contain a line of symmetry. Based on the main result in Albouy et al. (2008) (applicable to our problem only when $m > 0$), we might expect each solution to contain two lines of symmetry, forming a rhombus. We show here that this is not necessarily true. When $\lambda' > 0$ and $m \in (m^*, -1/2)$, where $m^* \approx -0.5951$, there exists a family of convex kite central configurations that are not rhombi. There does exist a family of rhombi when $\lambda' > 0$, as discussed in Sect. 7.4. In fact, this family of rhombi undergoes a pitchfork bifurcation at $m = m^*$ that results in the convex kites.

Proposition 7.5 *Let $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m < 0$. Any strictly planar solution to the Albouy–Chenciner equations with $\lambda' = 1$ must possess a line of symmetry.*

Proof The type of computations required are very similar to those used in Sects. 6.1 and 6.3. Consider the Albouy–Chenciner and the unsymmetrized Albouy–Chenciner equations together with the Cayley–Menger determinant and Dziobek equations with the normalization $\lambda' = 1$. Take a Gröbner basis of the ideal generated by such equations and saturate with respect to some of the variables to eliminate possible solutions where one of the mutual distances has zero length. Then, saturating with respect to the differences $(s_{13}-s_{24})$, $(s_{23}-s_{14})$, $(s_{13}-s_{14})$, $(s_{23}-s_{24})$, $(s_{13}-s_{23})$, and $(s_{14}-s_{24})$ yields a system of 12 polynomial equations. Analyzing this system by the method of triangular decomposition of semi-algebraic systems (with the inequalities $s_{ij} > 0$) yields an empty triangular system. Therefore, due to the saturation and the Symmetry Lemma, there are no asymmetric solutions in this case.   □

To find the solutions with $\lambda' > 0$, we impose the symmetry on the configuration and use Eq. (7) since the areas A_i are easily computable. Suppose that vortices 1 and 2 are on the axis of symmetry and set $r_{14} = r_{13}$ and $r_{24} = r_{23}$. We impose the

scaling $r_{34} = 1$ rather than choose $\lambda' = 1$. Introduce two new variables x and y measuring the distance between the intersection of the two diagonals and vertices 1 and 2, respectively. Thus, we have the simple geometric equations

$$x^2 + \frac{1}{4} = r_{13}^2, \qquad y^2 + \frac{1}{4} = r_{23}^2 \quad \text{and} \quad x + y = r_{12}. \tag{36}$$

Equations (36) taken together with $r_{34} = 1$ imply that the Cayley–Menger determinant e_{CM} vanishes.

Under this setup, convex configurations correspond to $x > 0$ and $y > 0$, while concave configurations have $xy < 0$. The oriented areas of the four triangles are given by $A_1 = y/2$, $A_2 = x/2$ and $A_3 = A_4 = -r_{12}/4$, where the signs are taken without loss of generality. Note that one of the Dziobek equations in (8) is automatically satisfied. The other equation yields an expression for λ' given by

$$\lambda' = \frac{r_{13}^2 r_{23}^2 - r_{12}^2}{r_{12}^2 (r_{13}^2 + r_{23}^2) - r_{13}^2 r_{23}^2 (r_{12}^2 + 1)}.$$

Equations (7), the formula for λ' and Eq. (36) yield a polynomial system in the variables $r_{12}, r_{13}, r_{23}, x, y, \lambda'$ and m. We then saturate this system with respect to $r_{13}, r_{23}, r_{13} - r_{23}$, and $r_{13} + r_{23}$ to eliminate the rhombus solutions and ensure that the mutual distance variables are nonzero. Denote the resulting polynomial system as P_{kite}. Analyzing Gröbner bases of the ideal generated by P_{kite} with different elimination orderings yields the following theorem. Recall that $m^* \approx -0.5951$ is the only real root of the cubic $9m^3 + 3m^2 + 7m + 5$.

Theorem 7.6 *For $m^* < m < -1/2$, there exist four convex kite configurations with vertices 1 and 2 on the axis of symmetry. These solutions have $\lambda' > 0$ and are not rhombi. There are no other strictly kite solutions (with vertices 1 and 2 on the axis of symmetry) with $\lambda' > 0$ and $m < 0$.*

Proof Let G_{kite} be the Gröbner basis of the ideal generated by P_{kite} with respect to the lex order where $r_{23} > r_{13} > r_{12} > \lambda' > y > x > m$. The first element in G_{kite} is an even eighth-degree polynomial in x with coefficients in m. Letting $z = x^2$, this polynomial is given by

$$\begin{aligned}
\zeta_m(z) = {}& 256m^2(m+2)(2m+1)^2 z^4 - 256m\bigl(9m^4 + 23m^3 + 17m^2 - m - 3\bigr)z^3 \\
& + \bigl(1728m^5 + 3136m^4 + 992m^3 - 384m^2 + 64m + 128\bigr)z^2 \\
& + \bigl(-432m^5 - 336m^4 + 48m^3 - 80m^2 + 16m + 64\bigr)z + (m+2)^3.
\end{aligned}$$

Computing a Gröbner basis with the same ordering of variables except that $x > y > m$ yields the same polynomial with $z = y^2$. The discriminant of ζ_m is a positive multiple of

$$m^4(m+1)^3(2m+1)\bigl(9m^3 + 3m^2 + 7m + 5\bigr)\bigl(9m^2 + 4m - 1\bigr)^2(m-1)^2\bigl(q_k(m)\bigr)^2,$$

where $q_k(m) = 2m^5 + 8m^3 + 14m^2 + 4m - 1$. Note that the discriminant vanishes at $m = m^*$. It also vanishes at $m = -1/2$, where $\zeta_m(z)$ becomes a cubic polynomial with roots $3/4$ and $-1/4$ (multiplicity 2). It is straightforward to check that the discriminant is strictly negative for $m^* < m < -1/2$. Consequently, $\zeta_m(z)$ has precisely two real roots for each m-value in this interval.

To see that these roots are always positive, we first use resultants to compute the repeated root of $\zeta_m(z)$ when $m = m^*$. This yields the value $z^* \approx 1.9566$, the only root of the cubic $c_u(z) = 320z^3 - 656z^2 + 60z - 3$. Next consider the ideal in $\mathbb{C}[z, m]$ generated by the system $\{\zeta_m, c_u\}$, which has (z^*, m^*) in its variety. Computing a Gröbner basis for this ideal which eliminates z, we obtain a polynomial in m that has no roots for $m^* < m \leq -1/2$. Since $\zeta_m(z^*) < 0$ when $m = -1/2$, it follows that $\zeta_m(z^*) < 0$ for any $m \in (m^*, -1/2)$. Then, since the constant term and leading coefficient of ζ_m are both positive for $m^* < m < -1/2$, the two real roots of ζ_m must be positive.

Choose x to be the positive square root of one of the roots of ζ_m. The second element in the Gröbner basis G_{kite} is linear in y, and the coefficient of y is nonzero for $m^* < m < -1/2$. Thus, by the Extension Theorem, we can extend a solution (x, m) of $\zeta_m(x) = 0$ to a unique partial solution (y, x, m), where y must be real. By considering the other polynomials in G_{kite}, three of which are given by Eq. (36), the Extension Theorem repeatedly applies to extend (y, x, m) to a unique full solution of system P_{kite} where $x, r_{12}, r_{13}, r_{23}$ are all positive and $r_{13} \neq r_{23}$. As expected, many of the elements in G_{kite} featuring the saturation variable for $r_{13} - r_{23}$ have a leading coefficient that is a multiple of the key cubic $9m^3 + 3m^2 + 7m + 5$.

To see that λ' is positive, we compute a Gröbner basis for P_{kite} which eliminates all variables except λ' and m. The first element in this Gröbner basis, a quadratic polynomial in λ' with coefficients in m, is

$$2m^2(m+1)\lambda'^2 + (4m-1)(m+1)^2\lambda' + m(m+2)(2m+1). \tag{37}$$

It is straightforward to show that the roots of this quadratic are real and positive for $m^* < m < -1/2$. Consequently, $\lambda' > 0$ and the configuration corresponding to the solution guaranteed by the Extension Theorem is convex. Therefore, $y > 0$ is also ensured, and by symmetry, y must be the positive square root of the remaining real root of ζ_m. Since we have saturated to eliminate the rhombus solutions, we must have $x \neq y$. Thus, there are two solutions: one where $x > \sqrt{z^*} > y > 0$, and one where $y > \sqrt{z^*} > x > 0$. The full count of four solutions comes from reflecting each of these solutions about the axis of symmetry (or simply interchanging vortices 3 and 4). This proves the first part of the theorem.

To see that there are no other solutions (other than a rhombus) with $\lambda' > 0$, first consider the case $m \in (-1/2, 0)$. According to Eq. (37), the two possible choices for λ' are real, with opposite sign. Denote the larger, positive root as λ'_+ and suppose there was a real solution to system P_{kite} with $\lambda' = \lambda'_+ > 0$. Since the quadratic (37) is strictly negative when $\lambda' = 3$ and $m \in (-1/2, 0)$, we can assume that $\lambda'_+ > 3$. Next, the top two equations in (7) imply that

$$4m^2xy(1 + \lambda') = 1 + \lambda'r_{12}^2. \tag{38}$$

If our real solution has $\lambda' > 0$, then it is convex and $xy > 0$. Since $\lambda'_+ > 3$, we have

$$1 + \lambda'_+ \left(x^2 + y^2\right) + xy\left(\lambda'_+ - 1\right) > 0.$$

Taking $m \in (-1/2, 0)$, this in turn implies that

$$4m^2 xy\left(1 + \lambda'_+\right) < xy\left(1 + \lambda'_+\right) < 1 + \lambda'_+(x + y)^2,$$

which violates Eq. (38). Therefore, any real solution for $m \in (-1/2, 0)$ must have $\lambda' < 0$.

Recall that, for $m = -1/2$, ζ_m reduces to a cubic polynomial with only one positive root at $\zeta = 3/4$. Therefore, the only possible solution to system P_{kite} at $m = -1/2$ has $x = y = \sqrt{3}/2$, a rhombus solution. At $m = m^*$, ζ_m has only one positive root at z^* (multiplicity two), which also implies that $x = y$. For the case $-1 < m < m^*$, the discriminant of ζ_m is positive; however, using Lemma 3.5 it is straightforward to check that all the roots of ζ_m are complex. Hence there are no real solutions to system P_{kite} when $-1 < m < m^*$.

For $m < -1$, the discriminant of ζ_m is negative so there are two real roots. To determine their sign, we note that $\zeta_m(-1/4) = 128m^3(m + 1)(2m + 1)$ is strictly negative for $-2 < m < -1$. Since both the leading coefficient and the constant term of ζ_m are positive when $-2 < m < -1$, it follows that the real roots of ζ_m are negative in this case, and there are no real solutions to system P_{kite}. For $m < -2$, the signs of the leading coefficient and constant term both flip to become negative. Since $\zeta_m(3/4) = -128(m - 1)(2m + 1)^2$ is strictly positive when $m < -2$, it follows that ζ_m has two positive real roots. However, examining the roots of the quadratic in Eq. (37), we see that $\lambda' < 0$ whenever $m < -2$. Finally, when $m = -2$, ζ_m reduces to a cubic with only one real root at zero. Thus, we have shown that, for $m < 0$ and $m \notin (m^*, -1/2)$, there are either no real solutions or only solutions with $\lambda' < 0$. This completes the proof of Theorem 7.6. □

Remark

1. Since Theorem 7.6 applies for all $m < 0$, we do not need to consider the case with vortices 3 and 4 on the axis of symmetry. If there existed a solution for some $m \in (-1, 0)$ having $\lambda' > 0$ and vortices 3 and 4 on the axis of symmetry, we could rescale the vortex strengths by $1/m$ and relabel the vortices to obtain a solution with $m < -1$, $\lambda' > 0$, and vortices 1 and 2 on the axis of symmetry. But this contradicts the last statement of the theorem.
2. For completeness, we note that the cases having real, positive solutions with $\lambda' < 0$ agree with the results in Sect. 7.2. For example, when $-1/2 < m < 0$, we find concave kite configurations with vortices 1 and 2 on the axis of symmetry, as predicted by Theorem 7.2. For $m < -2$, we find convex kite solutions with vortices 1 and 2 on the axis of symmetry. Rescaling the vortex strengths by $1/m$ and relabeling the vortices gives a family of convex kites with vortices 3 and 4 on the axis of symmetry and $-1/2 < m < 0$. This also concurs with Theorem 7.2.

7.4 The Rhombus Solutions

In this final section we study convex kite configurations that have two lines of symmetry, namely rhombus configurations. Here we have four congruent exterior sides, $r_{13} = r_{14} = r_{23} = r_{24}$ with the diagonals satisfying the relation $r_{12}^2 + r_{34}^2 = 4r_{13}^2$. The areas satisfy $A_1 = A_2 = -A_3 = -A_4$. The first and last equations in (9) are clearly satisfied. The middle four equations in (9) are equivalent. They, along with the value of λ' determined by (8), yield a formula for m given by

$$m = \frac{x^2(3 - x^2)}{3x^2 - 1}, \tag{39}$$

where $x = r_{34}/r_{12}$ is the ratio between the diagonals of the rhombus. Note that $m > 0$ if and only if $1/\sqrt{3} < x < \sqrt{3}$. As usual, we restrict to the case $-1 < m \le 1$.

Unlike the isosceles trapezoid case, here we have solutions for $m < 0$. In fact, there are two geometrically distinct families of rhombi when $m < 0$. This can be seen by inverting Eq. (39), which yields

$$x^2 = \frac{1}{2}\left(3 - 3m \pm \sqrt{(3 - 3m)^2 + 4m}\right). \tag{40}$$

Choosing $+$ in Eq. (40) yields a solution that exists for $-1 < m \le 1$. We will call this solution rhombus A. Taking $-$ in Eq. (40) yields a solution valid for $-1 < m < 0$. We will call this solution rhombus B.

Using Eq. (4), we compute that

$$\lambda = \frac{4(m^2 + 4m + 1)}{r_{12}^2(2 + 3m - 3m^2 \pm m\sqrt{9m^2 - 14m + 9})}, \tag{41}$$

where $+$ is used for rhombus A and $-$ is taken for rhombus B. Note that the numerator of (41) vanishes at $m = -2 + \sqrt{3} \approx -0.2679$. The denominator also vanishes at this special value, but only when $+$ is chosen. For the rhombus A solution, the value of the angular velocity λ is always positive (thus $\lambda' < 0$) and monotonically increasing in m. In contrast, for the rhombus B solution, we have $\lambda > 0$ (thus $\lambda' < 0$) only for $-2 + \sqrt{3} < m < 0$. At the special value $m = -2 + \sqrt{3}$, the rhombus B relative equilibrium becomes an equilibrium since $\lambda = 0$ (see Sect. 4.1). As m decreases through $-2 + \sqrt{3}$, the direction of rotation flips, λ becomes negative and λ' becomes positive. We summarize our conclusions in the following theorem.

Theorem 7.7 *There exist two one-parameter families of rhombi relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. The vortices 1 and 2 lie on opposite sides of each other, as do vortices 3 and 4. Let $\beta = 3 - 3m$. The mutual distances are given by*

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4m}\right) \quad and \quad \left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{8}\left(\beta + 2 \pm \sqrt{\beta^2 + 4m}\right), \tag{42}$$

describing two distinct solutions. Taking $+$ in (42) yields a solution called rhombus A that exists for $m \in (-1, 1]$ and always has $\lambda > 0$. Taking $-$ in (42) yields a solution

Fig. 7 A plot of x versus m for the two rhombus families of relative equilibria, where $x = r_{34}/r_{12}$ is the ratio between the diagonals of the rhombus. The *top curve* corresponds to rhombus A while the *bottom curve* refers to rhombus B. The × on the bottom curve indicates the location of the equilibrium point for rhombus B at $(m = -2 + \sqrt{3}, x = 2 - \sqrt{3})$. The relative equilibria to the right of this point rotate in the opposite direction as those to the left
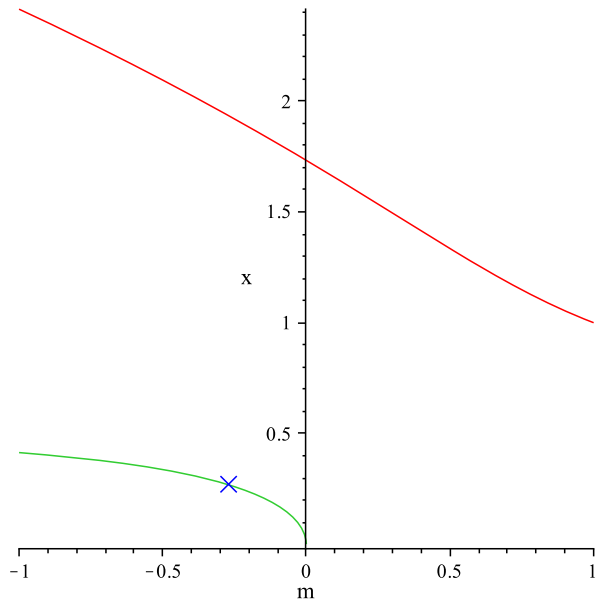

called rhombus B that exists for $m \in (-1, 0)$ and has $\lambda > 0$ for $m \in (-2 + \sqrt{3}, 0)$, but $\lambda < 0$ for $m \in (-1, -2 + \sqrt{3})$. At $m = -2 + \sqrt{3}$, rhombus B becomes an equilibrium. The case $m = 1$ reduces to the square. For the rhombus A family, the larger pair of vortices, (vortices 1 and 2), lie on the shorter diagonal, while for the rhombus B family, the larger pair lie on the longer diagonal.

Proof The formula for r_{13}/r_{12} comes from $1 + x^2 = 4(r_{13}/r_{12})^2$. It is easy to show from Eq. (40) that for rhombus A we have $x > 1$ for $-1 < m < 1$. Likewise, for rhombus B, we have $x < 1$ for $-1 < m < 0$. This verifies the last statement of the theorem. □

One way to visualize the two rhombus solutions is captured in Fig. 7, where a plot of $x = r_{34}/r_{12}$ versus m is shown. Beginning with the square at $m = 1$, as m decreases toward $m = 0$ the ratio of the diagonals for rhombus A increases from 1 to $\sqrt{3}$, while the ratio between the shorter diagonal and the common side length decreases from $\sqrt{2}$ to 1. At $m = 0$ a bifurcation occurs, and the rhombus B family is born emerging out of a binary collision between vortices 3 and 4, while the rhombus A family persists. The two families are distinguished by the fact that the larger pair of vortices, (vortices 1 and 2), lie on the shorter diagonal in rhombus A and on the longer diagonal in rhombus B (see Fig. 8). As $m \to -1^+$, $x \to \sqrt{2} - 1$ for rhombus B while $x \to \sqrt{2} + 1$ for rhombus A.

Remark

1. The rhombus B family undergoes a pitchfork bifurcation at the special parameter value $m^* \approx -0.5951$. As m increases through m^*, the rhombus bifurcates into
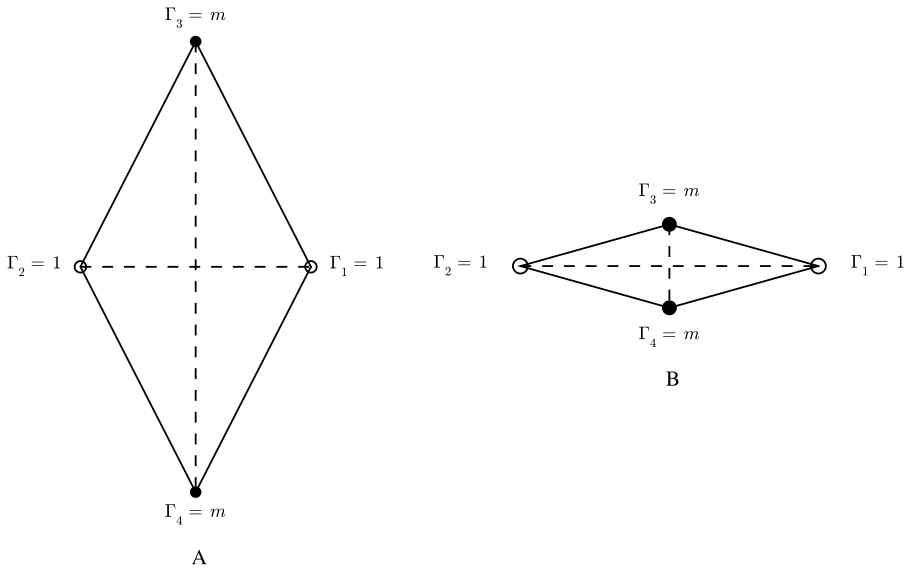
Fig. 8 The two distinct rhombi relative equilibria when $m = -0.3$. Rhombus A is to the left, rhombus B to the right. The solutions rotate in opposite directions

two convex kite solutions with the positive strength vortices on the axis of symmetry. The two kites are distinguished by whether $r_{13} > r_{23}$ or $r_{13} < r_{23}$. Since the rhombus solution continues to exist past the bifurcation, we have a pitchfork bifurcation. The null space of the Hessian matrix $D^2(H + \lambda I)$ for rhombus B, evaluated at $m = m^*$, is one-dimensional (excluding the "trivial" vector in the direction of rotation) and contains a vector corresponding to a perturbation in the direction of the convex kite solution found in Sect. 7.3.

2. The linear stability of the relative equilibria corresponding to the two rhombus families can be determined analytically (see Roberts 2013). It turns out that rhombus A is linearly stable for $-2 + \sqrt{3} < m \le 1$. It is degenerate at $m = -2 + \sqrt{3}$ and unstable for $-1 < m < -2 + \sqrt{3}$. The rhombus B family is always unstable, containing at least one pair of real eigenvalues. However, as m increases through m^*, one pair of eigenvalues changes from pure imaginary (stable) to real, which is precisely the type of behavior expected at a pitchfork bifurcation.

3. Using Eqs. (41) and (42), it is straightforward to check that parts 1 and 3 of Proposition 2.4 are satisfied for both rhombus families.

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