SVD computation example

Example: Find the SVD of A, $U\Sigma V^T$, where $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

First we compute the singular values σ_i by finding the eigenvalues of AA^T.

$$AA^T = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}.$$

The characteristic polynomial is $det(AA^T - \lambda I) = \lambda^2 - 34\lambda + 225 = (\lambda - 25)(\lambda - 9)$, so the singular values are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{9} = 3$.

Now we find the right singular vectors (the columns of V) by finding an orthonormal set of eigenvectors of A^TA. It is also possible to proceed by finding the left singular vectors (columns of U) instead. The eigenvalues of A^TA are 25, 9, and 0, and since A^TA is symmetric we know that the eigenvectors will be orthogonal.

For $\lambda = 25$, we have

$$A^TA - 25I = \begin{pmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{pmatrix}$$

which row-reduces to $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. A unit-length vector in the kernel of that matrix is $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$.

For $\lambda = 9$ we have $A^TA - 9I = \begin{pmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{pmatrix}$ which row-reduces to $\begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}$. A unit-length vector in the kernel is $v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}$.

For the last eigenvector, we could compute the kernel of A^TA or find a unit vector perpendicular to v_1 and v_2. To be perpendicular to $v_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ we need $-a = b$. Then the condition that $v_2^T v_3 = 0$ becomes $2a/\sqrt{18} + 4c/\sqrt{18} = 0$ or $-a = 2c$. So $v_3 = \begin{pmatrix} a \\ -a \\ -a/2 \end{pmatrix}$ and for it to be unit-length we need $a = 2/3$ so $v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}$.

So at this point we know that

$$A = U\Sigma V^T = U \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$

Finally, we can compute U by the formula $\sigma u_i = Av_i$, or $u_i = \frac{1}{\sigma} Av_i$. This gives $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$. So in its full glory the SVD is:

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$