4.2 Lecture 17: Outer Product, Adjoint operators

4.2.1 Outer product

Another convenient representation of linear operators is an outer product representation defined by,

\[ \langle \psi | \phi \rangle (\langle \phi' | ) \equiv \langle \psi | \phi' \rangle = \langle \phi' | \psi \rangle. \]

(4.15)

For example an identity operator can be represented as a sum of outer product of an orthonormal basis, i.e.

\[ \hat{I} = \sum_{i=1}^{n} |i\rangle \langle i| \]

(4.16)

This is known as a completeness relation which can be used to obtain an outer product representations of operators,

\[ \hat{A} = \hat{I} \hat{A} \hat{I} = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle i | \hat{A} | j \rangle |i \rangle \langle j|. \]

(4.17)

Eigenvectors \(|i\rangle\) and their respective eigenvalues \(\lambda_i\) of a linear operator \(\hat{A}\) are defined by

\[ \hat{A} |i\rangle = \lambda_i |i\rangle. \]

(4.18)

In a matrix representation the eigenvalues can be determined from a characteristic equation

\[ \det \left( \hat{A} - \lambda \hat{I} \right) = 0. \]

(4.19)

Diagonalizable representation of an operator (also known as an orthonormal decomposition) is given by

\[ \hat{A} = \sum_{i=1}^{n} \lambda_i |i\rangle \langle i| \]

(4.20)

where the eigenvectors \(|i\rangle\) form an orthonormal set.

4.2.2 Adjoint operators

For every linear operator \(\hat{A}\) on a Hilbert space there exist an adjoint (or Hermitian conjugate) operator defined as

\[ (|\psi\rangle, \hat{A} |\phi\rangle) \equiv (\hat{A}^\dagger |\psi\rangle, |\phi\rangle). \]

(4.21)
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It follows that

\[(AB)^\dagger = B^\dagger A^\dagger\]  \hspace{1cm} (4.22)

and

\[(\hat{A}|\psi\rangle)^\dagger = \langle\psi|\hat{A}^\dagger,\]  \hspace{1cm} (4.23)

where

\[|\psi\rangle^\dagger \equiv \langle\psi|\]  \hspace{1cm} (4.24)

In a matrix representation an adjoint operator is defined as

\[\hat{A}^\dagger \equiv (\hat{A}^*)^T\]  \hspace{1cm} (4.25)

where \((\cdot)^*\) is a complex conjugation and \((\cdot)^T\) is a transpose operation.

Some useful definitions:

- \(\hat{A}\) is a **positive definite** operator if \((|\psi\rangle, \hat{A}|\psi\rangle)\) is a positive real number for all \(|\psi\rangle \neq 0\).
- \(\hat{A}\) is a **self-adjoint** (or Hermitian) operator if \(\hat{A}^\dagger = \hat{A}\). Any positive definite operator is Hermitian.
- \(\hat{A}\) is a **normal** operator if \(\hat{A}\hat{A}^\dagger = \hat{A}^\dagger \hat{A}\). For example, any Hermitian operator is normal.
- \(\hat{P}\) is a **projection** operator if \(\hat{P} = \sum_{i=1}^{k} |i\rangle\langle i|\), where \(|i\rangle\) is an orthonormal basis and \(k \leq n\).
- \(\hat{U}\) is a **unitary** operator if \(\hat{U}^\dagger \hat{U} = \hat{I}\). Any pair of orthonormal basis \(|\psi_i\rangle\) and \(|\varphi_i\rangle\) can be used to define unitary operators, i.e. \(\hat{U} = \sum_{i=1}^{n} |\psi_i\rangle\langle \varphi_i|\).

Some useful results:

- Unitary operators preserve the inner product,
  \[\langle \hat{U}|\psi\rangle, \hat{U}|\varphi\rangle\rangle = \langle \psi|\hat{U}^\dagger \hat{U}|\varphi\rangle = \langle \psi|\hat{I}|\varphi\rangle = \langle \psi|\varphi\rangle.\]  \hspace{1cm} (4.26)

- Normal operators have a diagonal representation,
  \[\hat{A} = \sum_{i=1}^{n} \lambda_i |i\rangle\langle i|.\]  \hspace{1cm} (4.27)

It follows that the positive definite operators as well as Hermitian operators have diagonal representations.

**Homework problem:** Prove the spectral decomposition theorem which states that any normal operator is diagonal with respect to some orthonormal basis.
4.2.3 Operator functions

If \( \hat{A} \) has a diagonal decomposition (4.27), then

\[
f(\hat{A}) \equiv \sum_{i=1}^{n} f(\lambda_i) |i\rangle\langle i| \tag{4.28}
\]

for an arbitrary function \( f \).

Another important matrix function is the *trace* defined as

\[
\text{tr}(\hat{A}) \equiv \sum_{i=1}^{n} A_{ii} \tag{4.29}
\]

and has the following properties:

\[
\begin{align*}
\text{tr}(\hat{A}\hat{B}) &= \text{tr}(\hat{B}\hat{A}) \\
\text{tr}(\hat{A} + \hat{B}) &= \text{tr}(\hat{A}) + \text{tr}(\hat{B}) \\
\text{tr}(z\hat{A}) &= z \text{tr}(\hat{A}).
\end{align*}
\tag{4.30-4.32}
\]

These imply an invariance under similarity transformations

\[
\text{tr}\left(\hat{U}A\hat{U}^\dagger\right) = \text{tr}\left(\hat{U}^\dagger\hat{U}A\right) = \text{tr}\left(A\right). \tag{4.33}
\]

Because of this invariance one can define a trace of an operator as a trace of any of its matrix representations.