Chapter 8

Conservation Laws

8.1 Conservation of Charge and Energy

As was already shown, the continuity equation

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot J = 0, \]  

(8.1)

can be derived by considering the flow of charges from a given volume,

\[ \frac{\partial Q}{\partial t} = \frac{\partial}{\partial t} \int \rho(r) d^3r = - \int J(r) \cdot da = - \int \nabla \cdot J(r) d^3r \]  

(8.2)

The continuity of charges expresses the fact that the total charge is conserved.

Another conservation law can be derived from the Faraday’s and Ampère’s laws

\[ \nabla \times E + \frac{\partial B}{\partial t} = 0 \]  

(8.3)

\[ \nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} = \mu_0 J. \]  

(8.4)

By differentiating the energy density (7.90) with respect to time we get that

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) \right) = \epsilon_0 E \cdot \left( \frac{1}{\epsilon_0 \mu_0} \nabla \times B - \frac{1}{\epsilon_0 \mu_0} \mu_0 J \right) + \frac{1}{\mu_0} B \cdot (\nabla \times E) \]  

(8.5)

or

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) \right) = - \frac{1}{\mu_0} \nabla \cdot (E \times B) - J \cdot E. \]  

(8.6)

This is the Poynting’s theorem which expresses the fact that energy is conserved.
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Indeed the energy required to assemble a static distribution of charges is given by (7.89) and the energy required to assemble a distribution of currents is given by (7.90). Then from the Poynting’s theorem (8.6), the rate of change of energy

\[
\frac{\partial u}{\partial t} = \int \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d^3r
\]

\[= - \int \frac{1}{\mu_0} (E \times B) \cdot da - \int J \cdot E d^3r. \tag{8.7}\]

Note that the second term is the rate at which the total external work is done on all charges

\[
\frac{\partial W}{\partial t} = \frac{\partial}{\partial t} \int \rho (E + v \times B) \cdot v dt d^3r.
\]

\[= \int \rho (E + v \times B) \cdot v d^3r
\]

\[= \int E \cdot (\rho v) d^3r
\]

\[= \int E \cdot J d^3r. \tag{8.8}\]

If the system is closed (i.e. there are no external work), then the Poynting’s theorem (8.6) can be written as

\[
\frac{\partial u}{\partial t} = -\nabla \cdot S \tag{8.9}
\]

where

\[u = \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \tag{8.10}\]

is the energy density and

\[S = \frac{1}{\mu_0} E \times B \tag{8.11}\]

is the Poynting vector. What it describes is the energy flux similarly to how the current density describes the flux of charges.

For example, consider a current \(I\) flowing down the wire which heats the environment with a power that we are to calculate. The electric field in the wire is

\[E = \frac{V}{L} \hat{z} \tag{8.12}\]

and the magnetic field at the surface is

\[B = \frac{\mu_0 I}{2\pi a} \hat{\phi} \tag{8.13}\]
and therefore the Poynting vector is pointing towards the center of the wire

\[ S = \frac{1}{\mu_0} E \times B. \]  

(8.14)

Then we can use the Poynting’s theorem to find the power

\[
\int \frac{\partial u}{\partial t} d^3r = - \int \nabla \cdot S d^3r = - \int S \cdot da = \frac{VI}{2\pi aL} 2\pi aL = VI \]

(8.15)

in agreement with (7.85).

### 8.2 Conservation of Momentum

When one considers motion of charged particles in electromagnetic field very often one encounters situations when the linear or angular momentum of particles is not conserved. The problem is that in addition to momentum carried by particle the momentum can also be carried by fields such as electric and magnetic fields.

Consider the total electromagnetic force in some volume

\[
F = \int_V \rho \left( E + v \times B \right) d^3r
= \int_V \left( \rho E + J \times B \right) d^3r
\]

(8.16)

Using two Maxwell’s equations (i.e. \( \epsilon_0 \nabla \cdot E = \rho \) and \( \frac{1}{\mu_0} \nabla \times B - \epsilon_0 \frac{\partial B}{\partial t} = J \)) the force density can be rewritten as

\[
f = \frac{\partial S}{\partial t} = \epsilon_0 \left( \nabla \cdot E \right) E + \left( \frac{1}{\mu_0} \nabla \times B - \epsilon_0 \frac{\partial B}{\partial t} \right) \times B. \]

(8.17)

and the third Maxwell equation (i.e. \( \nabla \times E + \frac{\partial B}{\partial t} = 0 \)) can used to rewrite the time derivative of Poynting vector as

\[
\mu_0 \frac{\partial S}{\partial t} = \frac{\partial}{\partial t} \left( E \times B \right)
= \frac{\partial E}{\partial t} \times B + E \times \frac{\partial B}{\partial t}
= \frac{\partial E}{\partial t} \times B - E \times (\nabla \times E) \]

(8.18)
or
\[ \epsilon_0 \frac{\partial E}{\partial t} \times B = \mu_0 \epsilon_0 \frac{\partial S}{\partial t} + \epsilon_0 E \times (\nabla \times E). \] (8.19)

By substituting (8.19) into (8.17) and using the fourth Maxwell equation (i.e. \( \nabla \cdot B = 0 \)) we obtain
\[
f = \epsilon_0 (\nabla \cdot E) E - \frac{1}{\mu_0} B \times (\nabla \times B) - \epsilon_0 \frac{\partial E}{\partial t} \times B
\]
\[
= \epsilon_0 [(\nabla \cdot E) E - E \times (\nabla \times E)] + \frac{1}{\mu_0} [(\nabla \cdot B) B - B \times \nabla \times B] - \mu_0 \epsilon_0 \frac{\partial S}{\partial t}
\]
This can be modified further using
\[
\nabla E^2 = 2 (E \cdot \nabla) E + 2 E \times (\nabla \times E) \quad (8.21)
\]
\[
\nabla B^2 = 2 (B \cdot \nabla) B + 2 B \times (\nabla \times B) \quad (8.22)
\]
to obtain
\[
f = \epsilon_0 \left[ (\nabla \cdot E) E + (E \cdot \nabla) E - \frac{1}{2} \nabla E^2 \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot B) B + (B \cdot \nabla) B - \frac{1}{2} \nabla B^2 \right] - \mu_0 \epsilon_0 \frac{\partial S}{\partial t}. \quad (8.23)
\]

Now if we define the Maxwell stress tensor (spatial components of the so-called energy-momentum or stress-energy tensors)
\[
T_{ij} \equiv \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)
\]
\[
= \epsilon_0 \left( \frac{1}{2} (E_1^2 - E_2^2 - E_3^2) \quad E_1 E_2 \quad E_1 E_3 \right)
\]
\[
\left( \frac{1}{2} (E_2^2 - E_3^2 - E_1^2) \quad -E_2^2 + E_3^2 - E_1^2 \quad E_2 E_3 \right)
\]
\[
\left( \frac{1}{2} (E_3^2 - E_1^2 - E_2^2) \quad -E_3^2 + E_1^2 - E_2^2 \quad E_3 E_1 \right)
\]
\[
\frac{1}{\mu_0} \left( \frac{1}{2} (B_1^2 - B_2^2 - B_3^2) \quad B_1 B_2 \quad B_1 B_3 \right)
\]
\[
\left( \frac{1}{2} (B_2^2 - B_3^2 - B_1^2) \quad -B_2^2 + B_3^2 - B_1^2 \quad B_2 B_3 \right)
\]
\[
\left( \frac{1}{2} (B_3^2 - B_1^2 - B_2^2) \quad -B_3^2 + B_1^2 - B_2^2 \quad B_3 B_1 \right)
\]
then the force density can be expressed as
\[
f_i = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} T_{ji} - \mu_0 \epsilon_0 \frac{\partial S_i}{\partial t} \quad (8.25)
\]
or using the vector notation
\[
f = \nabla \cdot \mathbf{T} - \mu_0 \epsilon_0 \frac{\partial \mathbf{S}}{\partial t} \quad (8.26)
\]
where the divergence is taken with respect to either first or second index in $T_{ij}$ would not make a difference due to symmetry of the stress tensor

$$T_{ij} = T_{ji}. \tag{8.27}$$

This equation represents the flow of momentum that can be exchanged between charged particles and electric and magnetic field without violating the momentum conservation. We can now go back to the volume integral to express the external force as

$$F = \int_V \mathbf{f} \, d^3r = \int_S \mathbf{T} \cdot d\mathbf{a} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_V \mathbf{S} \, d^3r. \tag{8.28}$$

and in the static case as

$$F = \int_S \mathbf{T} \cdot d\mathbf{a}. \tag{8.29}$$

Thus, $\mathbf{T}$ is nothing but a force per unit area, or stress tensor whose diagonal terms describe pressure and off-diagonal terms describe shear.

For example, consider a hemisphere of a uniformly charged sphere of radius $R$ and charge $Q$. Since this is a static case the stress tensor (a local quantity) calculated over surface should be sufficient to calculate the net force on the hemisphere,

$$F = \int_{disk} \mathbf{T} \cdot d\mathbf{a} + \int_{bowl} \mathbf{T} \cdot d\mathbf{a}$$

$$= - \int_0^{2\pi} d\phi \int_0^R rdr \mathbf{T} \cdot \hat{z} + \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta d\theta R^2 \mathbf{T} \cdot \hat{r} \tag{8.30}$$

In Cartesian coordinates the radial unit vector is

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \tag{8.31}$$

and the relevant components of the stress tensor on the “bowl” are given by

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 (\rho R)^2 \sin \theta \cos \theta \cos \phi \tag{8.32}$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 (\rho R)^2 \sin \theta \cos \theta \sin \phi \tag{8.33}$$

$$T_{zz} = \frac{\epsilon_0}{2} (E_z^2 - E_x^2 - E_y^2) = \frac{\epsilon_0}{2} (\rho R)^2 (\cos^2 \theta - \sin^2 \theta) \tag{8.34}$$

where

$$\rho = \frac{Q}{4\pi \varepsilon_0 R^3} \tag{8.35}$$
or
\[
(T \cdot \hat{r})_z = T_{xx} \cdot da_x + T_{zy} \cdot da_y + T_{zz} \cdot da_z
\]
\[
= \epsilon_0 (\rho R)^2 (\sin^2 \theta \cos^2 \phi \cos \theta + \sin^2 \phi \cos^2 \theta + \frac{1}{2} \cos^3 \theta - \frac{1}{2} \sin^2 \theta \cos \theta)
\]
\[
= \frac{\epsilon_0}{2} (\rho R)^2 \cos \theta.
\]
\[
(8.36)
\]
Similarly on the "disk"
\[
(T \cdot \hat{z})_z = T_{zz} = \frac{\epsilon_0}{2} (E_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2} (\rho r)^2
\]
\[
(8.37)
\]
and thus,
\[
F_z = \int_0^{2\pi} d\phi \int_0^R rd\theta (\frac{\epsilon_0}{2} (\rho r)^2) + \int_0^{\pi/2} \sin \theta d\theta R^2 \frac{\epsilon_0}{2} (\rho R)^2 \cos \theta
\]
\[
= \epsilon_0 \rho^2 \pi \int_0^R r^3 dr + \epsilon_0 \rho^2 \pi R^4 \int_0^{\pi/2} \sin \theta d\theta
\]
\[
= \epsilon_0 \rho^2 \pi R^4 \left( \frac{1}{4} + \frac{1}{4} \int_0^{\pi/2} \sin(2\theta)d(2\theta) \right)
\]
\[
= \frac{3}{4} \epsilon_0 \left( \frac{Q}{4\pi \epsilon_0 R^2} \right)^2 \pi R^4
\]
\[
= \frac{1}{4\pi \epsilon_0} \frac{3Q^2}{16R^2}.
\]
\[
(8.38)
\]
Note that the stress tensor can be generalized to the stress-energy (or energy momentum) tensor
\[
T_{\mu \nu} = \begin{pmatrix}
  u & \mu_0 \epsilon_0 S_1 & \mu_0 \epsilon_0 S_2 & \mu_0 \epsilon_0 S_3 \\
  \mu_0 \epsilon_0 S_1 & -T_{11} & -T_{12} & -T_{13} \\
  \mu_0 \epsilon_0 S_2 & -T_{21} & -T_{22} & -T_{23} \\
  \mu_0 \epsilon_0 S_3 & -T_{31} & -T_{32} & -T_{33}
\end{pmatrix}
\]
\[
(8.39)
\]
and then the conservations of momentum and energy can be combined in a single covariant equation
\[
f^\mu + \nabla_\mu T^{\mu \nu} = 0
\]
\[
(8.40)
\]
where $f^\mu$ is the external four-force of the change of external (let say mechanical) momentum per unit time. The momentum stored in electromagnetic field or in the momentum density
\[
g = \mu_0 \epsilon_0 S = \epsilon_0 (E \times B).
\]
\[
(8.41)
\]
In the absence of the external momentum (i.e. $f = 0$) the conservation of momentum implies
\[ \int_S \mathbf{T} \cdot d\mathbf{a} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_V \mathbf{S} \cdot d\mathbf{r} \]  
(8.42)
or in a differential form
\[ \frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \mathbf{T}. \]  
(8.43)

Moreover, one can also define the angular momentum density
\[ \mathbf{l} = \mathbf{r} \times \mathbf{g} = \epsilon_0 [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] \]  
(8.44)
that must also be conserved. For example consider a long solenoid with current $I$, radius $R$ and $n$ turns per unit length and two coaxial cylindrical conducting shells of length $l$ and radii $a < R$ and $b > R$ with charges $Q$ and $-Q$ respectively. When the current is switched off the cylindrical shells start to rotate due to flow of angular momentum that was initially stored in electromagnetic fields. Initially the electric field between shells was
\[ \mathbf{E} = \frac{Q}{2\pi \epsilon_0 l} \frac{1}{s} \hat{s} \]  
(8.45)
and the magnetic field inside solenoid was
\[ \mathbf{B} = \mu_0 n I \hat{z}. \]  
(8.46)
This corresponds to momentum density
\[ \mathbf{g} = \epsilon_0 (\mathbf{E} \times \mathbf{B}) = -\frac{\mu_0 n I Q}{2\pi l s} \hat{\phi} \]  
(8.47)
and angular momentum density
\[ \mathbf{l} = \frac{\mu_0 n I Q}{2\pi l} \mathbf{s} \times \hat{\phi} = -\frac{\mu_0 n I Q}{2\pi l} \hat{\mathbf{z}} \]  
(8.48)
in the region $a < s < R$ or the total angular momentum
\[ \mathbf{L} = -\frac{\mu_0 n I Q}{2\pi l} \hat{\mathbf{z}} \int_a^R s ds \int_0^{2\pi} d\phi \int_0^l dz \hat{\mathbf{z}} \]
\[ = -\frac{1}{2} \mu_0 n I Q (R^2 - a^2) \hat{\mathbf{z}}. \]  
(8.49)
This angular momentum went into the angular momentum of charged cylinders due to induced electric field (Faraday’s law),
\[ \mathbf{E} = \begin{cases} 
-\frac{1}{2} \mu_0 n \frac{d}{dx} \frac{R^2}{s} \hat{\phi} & \text{for } s > R \\
-\frac{1}{2} \mu_0 n \frac{d}{dx} \hat{\phi} & \text{for } s < R.
\end{cases} \]  
(8.50)
Then the total torque on the cylinders is

\[
\mathbf{N}_b = \mathbf{r} \times (-Q \mathbf{E}) = \frac{1}{2} \mu_0 n Q \frac{dI}{dt} R^2 \mathbf{b} \times \hat{\phi} = \frac{1}{2} \mu_0 n Q R^2 \frac{dI}{dt} \hat{z}
\]  

(8.51)

and

\[
\mathbf{N}_a = \mathbf{r} \times (Q \mathbf{E}) = -\frac{1}{2} \mu_0 n Q \frac{dI}{dt} a \mathbf{a} \times \hat{\phi} = -\frac{1}{2} \mu_0 n Q a^2 \frac{dI}{dt} \hat{z}
\]  

(8.52)

and the total angular momentum

\[
\mathbf{L} = \int_0^R \frac{1}{2} \mu_0 n Q (R^2 - a^2) \frac{dI}{dt} \hat{z} = -\frac{1}{2} \mu_0 n I Q (R^2 - a^2) \hat{z}.
\]  

(8.53)

The angular momentum that was initially stored in the electric and magnetic field (8.49) is now in the angular momentum of the charges on the cylinders (8.53).