Chapter 12

Electrodynamics and Relativity

In Newtonian physics the three spatial dimensions $x, y$ and $z$ are connected by coordinate transformations, yet the time coordinate $t$ was always treated separately. For example one can rotate the Cartesian coordinate system without altering the distance

$$ (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2. \quad (12.1) $$

There are six linearly independent transformations of space: three shifts and three rotations. However, until special theory of relativity there were no useful definitions of invariant distance in space-time described by all four coordinates $t, x, y$ and $z$. We knew that the time coordinate must be different from space, but the connection was not clear until the following proposal for the invariant distance was made

$$ (\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2. \quad (12.2) $$

It is more convenient to set $c = 1$ and to use the upper indices notation

$$ t \to x^0 \quad (12.3) $$
$$ x \to x^1 \quad (12.4) $$
$$ y \to x^2 \quad (12.5) $$
$$ z \to x^3. \quad (12.6) $$

(Note that upper indices should not to be confused with exponentiating!) Then with the Einstein summation convention (i.e. always sum over repeated upper and lower indices) we obtain

$$ (\Delta s)^2 = \sum_{\mu, \nu = 0,1,2,3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \equiv \eta_{\mu\nu} \Delta x'^\mu \Delta x'^\nu, \quad (12.7) $$
where
\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (12.8)

What are the transformation which leave the distance in space-time invariant? For example if we change
\[ x^\mu \rightarrow x'^\mu + a^\mu \] (12.9)
the invariant distance defined by (12.9) would not change. These are the shifts in space \( a^1, a^2, a^3 \) and in time \( a^0 \). What about other transformations analogous to rotations? Consider
\[ x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\nu'} x'^\nu \] (12.10)
where \( \Lambda \) is some 4 \( \times \) 4 matrix such that in matrix notation \( x' = \Lambda x \). For the distance (12.7) to be invariant we must have
\[ (\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x') = (\Delta x')^T \Lambda^T \eta \Lambda (\Delta x) \] (12.11)
and thus,
\[ \eta = \Lambda^T \eta \Lambda \] (12.12)
or
\[ \eta_{\rho\sigma} = \Lambda^\mu_{\rho} \Lambda^\nu_{\sigma} \eta_{\mu\nu}. \] (12.13)
Such transformations are known as Lorentz transformations and the group such transformations is called Lorentz group or \( O(3,1) \).

There are six generators of \( O(3,1) \): the three usual rotations and three Lorentz boosts. For example,
\[ \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (12.14)
describes rotation where the angle \( \theta \in [0, \pi] \)
\[ \Lambda = \begin{pmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (12.15)
describes boosts where the parameter \( \phi \in (-\infty, +\infty) \). Nevertheless, it is still useful to think of boosts as rotations between space and time. Altogether how many linearly independent continuous transformations leave the distance (12.7) invariant? one time shift + three space shifts + three rotations + three boosts = ten! the group of such transformation is a non-abelian group known as the Poincare group. There are also discrete reflections of each of four coordinates, but they are not a part of the Poincare group.

The new ingredient in special relativity (that you must have seen before) are the boosts which correspond to changing coordinates to a moving frame. From (12.15) the transformed coordinates are

\[
\begin{align*}
t' &= t \cosh \phi - x \sinh \phi \\
x' &= -t \sinh \phi + x \cosh \phi.
\end{align*}
\]

Thus, in the original coordinate system the point corresponding to \( x' = 0 \) or

\[ -t \sinh \phi + x \cosh \phi = 0 \]

is moving with velocity

\[ v \equiv \frac{x}{t} = \tanh \phi. \]

Then the transformation laws (12.16) and (12.17) can be written in a more familiar form

\[
\begin{align*}
t' &= t \cosh (\tanh^{-1} v) - x \sinh (\tanh^{-1} v) = \gamma (t - vx) \\
x' &= -t \sinh (\tanh^{-1} v) + x \cosh (\tanh^{-1} v) = \gamma (x - vt)
\end{align*}
\]

where \( \gamma = 1/\sqrt{1 - v^2} \), but you should check this.

It is also easy to see what the boost transformation do to your axes \( x = 0 \) and \( y = 0 \) using the space-time diagram. The transformed axes \( x' = 0 \) and \( t' = 0 \) in terms of the old coordinates are described by lines

\[
\begin{align*}
x &= t \tanh \phi \\
x &= \frac{t}{\tanh \phi}
\end{align*}
\]

The only paths that are the same in both coordinates systems are the ones described by light rays, i.e. \( x = \pm t \) is the same as \( x' = \pm t' \). The set of all light rays emitted from a given space-time point is called the light cone. This set divides all of the points into light-like (or null) separated, space-like separated, future time-like separated and past time-like separated.
12.1 Field tensor

Let us work in natural units such that Maxwell equations are given by

\[ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \]  
\[ \nabla \cdot \mathbf{E} = \rho \]  
\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]  
\[ \nabla \cdot \mathbf{B} = 0 \]

Then it is convenient to define an antisymmetric \((0, 2)\) electromagnetic field tensor,

\[ F_{\mu \nu} = -F_{\nu \mu} \]  

whose component are made out of the electric and magnetic fields

\[ F_{\mu \nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \]

with a Hodge dual

\[ \star F_{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \lambda \sigma} F_{\lambda \sigma} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} \]

where the Einstein summation convention (i.e. sum over repeated indices) is used. The upper indices are defined using the inverse metric tensor

\[ \eta^{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

For example

\[ F^{\mu \nu} = \eta^{\mu \lambda} \eta^{\nu \sigma} F_{\lambda \sigma} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \]

\[ \star F^{\mu \nu} = \eta^{\mu \lambda} \eta^{\nu \sigma} \star F_{\lambda \sigma} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} \]
Both $F$ and $\star F$ transforms under the Lorentz transformations exactly as a second rank tensor should, i.e.

\[
F_{\mu\nu} \rightarrow F'_{\mu'\nu'} = \Lambda_{\mu'}^{\mu} \Lambda_{\nu}^{\nu'} F_{\mu'\nu'}
\]

\[
F_{\mu\nu} \rightarrow \star F'_{\mu'\nu'} = \Lambda_{\mu'}^{\mu} \Lambda_{\nu}^{\nu'} \star F_{\mu'\nu'}.
\]

If we also define a four-vector current from an electric current $J$ and charge $J^0$ densities, i.e.

\[
J^\mu = \begin{pmatrix}
\rho \\
J_1 \\
J_2 \\
J_3
\end{pmatrix},
\]

then the Maxwell equations can be written in terms of components

\[
\epsilon_{ijk} \partial_j B_k - \partial_0 E_i = J_i 
\]

\[
\partial_i E_i = J_0
\]

\[
\epsilon_{ijk} \partial_j E_k + \partial_0 B_i = 0
\]

\[
\partial_i B_i = 0
\]

or in terms of field tensor

\[
\partial_j F^{ij} + \partial_0 F^{i0} = J^i
\]

\[
\partial_j F^{0j} = J^0
\]

\[
\partial_j (\star F^{ij}) + \partial_0 (\star F^{i0}) = 0
\]

\[
\partial_j (\star F^{0j}) = 0.
\]

This can be rewritten as only two equations

\[
\partial_\mu F^{\nu\mu} = J^\nu
\]

\[
\partial_\mu \star F^{\nu\mu} = 0.
\]

One can also use the Lorentz symmetry to derive an expression for the **Lorentz force** on a particle of charge $q$ and four-velocity $v^\mu$ is a four-vector given by

\[
f^\mu = q v^\nu F_{\nu}^\mu
\]

which reduces to the familiar expression from small velocities

\[
f = q (E + v \times B)
\]

as one would expect.
12.2 Variational Principle

In classical mechanics an action of a single particle described by coordinates $q(t)$ is

$$S[q] = \int dt \, L(q, \dot{q}). \quad (12.45)$$

which can be thought of as a functional of $q(t)$. Then the classical equations of motion can be obtained from the most important principle in physics—the variational principle:

$$\frac{\delta S[q]}{\delta q} = 0. \quad (12.46)$$

By Taylor expanding the Lagrangian

$$L(q + \delta q) \approx L(q) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \quad (12.47)$$

we obtain

$$0 = \frac{\delta S}{\delta q} = \int dt \left( \frac{\delta L}{\delta q} \right)$$

$$= \int dt \left( (L(q + \delta q) - L(q)) \delta q \right)$$

$$= \int dt \left( \frac{\partial L(q)}{\partial q} \delta q + \frac{\partial L(q)}{\partial \dot{q}} \delta \dot{q} \right)$$

$$= \int dt \left( \frac{\partial L(q)}{\partial \dot{q}} \delta \dot{q} + \frac{d}{dt} \left( \frac{\partial L(q)}{\partial \dot{q}} \right) \delta q \right)$$

or up to a boundary term

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \quad (12.49)$$

For example,

$$L = \frac{1}{2} q^2 - V(q) \quad (12.50)$$

gives rise to

$$\ddot{q} = - \frac{\partial V}{\partial q}. \quad (12.51)$$

In classical field theory an action for a collection of fields described by $\Phi^i(x^0, x^1, x^2, x^3)$ is

$$S[\Phi^1, ..., \Phi^N] = \int d^4x \, \mathcal{L}(\Phi^1, \partial_0 \Phi^1, \partial_1 \Phi^1, \partial_2 \Phi^1, \partial_3 \Phi^1, ...) \quad (12.52)$$
(which is a slightly more complicated functional) one can still use the variational principle to obtain $N$ equations of motion

$$\frac{\delta S}{\delta \Phi^i} = 0 \quad (12.53)$$

for $N$ degrees of freedom. Note that the action is dimensionless which suggests that the so-called Lagrangian density $\mathcal{L}$ must have the dimensions

$$[\mathcal{L}] = [\text{Length}]^{-4} = [\text{Time}]^{-4} = [\text{Mass}]^4 = [\text{Energy}]^4. \quad (12.54)$$

By Taylor expanding the perturbed Lagrangian

$$\mathcal{L}(\ldots, \Phi^i + \delta \Phi^i, \partial \mu \Phi^i, \partial \nu \Phi^i, \ldots) = \mathcal{L}(\ldots, \Phi^i, \partial \mu \Phi^i, \ldots) + \frac{\partial \mathcal{L}}{\partial \Phi^i} \delta \Phi^i + \frac{\partial \mathcal{L}}{\partial (\partial \mu \Phi^i)} \delta (\partial \nu \Phi^i) \quad (12.55)$$

we get (up to the boundary term)

$$0 = \frac{\delta S}{\delta \Phi^i} = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial \mu \left( \frac{\partial \mathcal{L}}{\partial (\partial \mu \Phi^i)} \right) \right] \delta \Phi^i \quad (12.56)$$

or

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial \mu \left( \frac{\partial \mathcal{L}}{\partial (\partial \mu \Phi^i)} \right) = 0. \quad (12.57)$$

For example, the scalar field Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial \mu \phi \partial \nu \phi - V(\phi) \quad (12.58)$$

gives rise to an equation of motion

$$\Box \phi - \frac{\partial V}{\partial \phi} = 0, \quad (12.59)$$

where

$$\Box \equiv \partial \mu \partial ^\mu. \quad (12.60)$$

For a massless vector field coupled to a conserved current, i.e.

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + A_\mu j^\mu \quad (12.61)$$

where

$$F_{\mu \nu} = \partial \mu A_\nu - \partial \nu A_\mu \quad (12.62)$$

the equations of motion are

$$\partial \mu F^{\mu \nu} = j^\nu. \quad (12.63)$$
12.3 Neother’s Theorem

Consider an infinitesimal transformation of the scalar field

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x). \quad (12.64)$$

We call this transformation a symmetry if the equations of motion are unchanged, i.e.

$$\left[ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right] (\phi) = \left[ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right] (\phi'). \quad (12.65)$$

This is certainly the case if the action is unchanged, i.e.

$$S[\phi] = S[\phi'] \quad (12.66)$$

but more generally there might be a boundary term since its presence would not change equations of motion.

If the Lagrangian density transforms as

$$L(x) \rightarrow L'(x) = L(x) + \alpha \partial_\mu J^\mu(x), \quad (12.67)$$

for some $J^\mu$, then the action would only gain a extra boundary term

$$S'[\phi] = \int d^4 x L'(x) =$$

$$= \int d^4 x \left( L(x) + \alpha \partial_\mu J^\mu(x) \right)$$

$$= \int d^4 x L'(x) + \oint J \cdot da = S[\phi] + \text{boundary term} \quad (12.68)$$

which would not modify the equations of motion. On the other hand

$$(L(\partial_\mu \phi + \alpha \partial_\mu \Delta \phi, \phi + \alpha \Delta \phi) - L(\partial_\mu \phi, \phi)) = \frac{\partial L}{\partial \phi} (\alpha \Delta \phi) + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi) \quad (12.69)$$

$$\alpha \partial_\mu J^\mu(x) = \alpha \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \left( \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right)$$

and due to equations of motion (12.57)

$$\partial_\mu J^\mu(x) = \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right) \quad (12.71)$$

or

$$\partial_\mu j^\mu = 0 \quad (12.72)$$
where

\[ j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi - J^\mu (x) \]  

(12.73)

This means that there must exist a conserved current \( j^\mu \) for every continues symmetry transformation, or equivalently there is a conserved charge density

\[ \rho \equiv j^0 \]  

(12.74)

and current

\[ \mathbf{J} \equiv (j^1, j^2, j^3) \]  

(12.75)

that satisfy the continuity equation

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}. \]  

(12.76)

A trivial example is a conserved current for the theory with

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi. \]  

(12.77)

Then the Lagrangian density, and thus the action and thus the equations of motion would not change is we change

\[ \phi \to \phi' = \phi + \alpha. \]  

(12.78)

In terms of general transformations (12.64) this corresponds to \( \Delta \phi = 1 \) and thus from equation (12.73) we get

\[ j^\mu = \partial_\mu \phi. \]  

(12.79)

A bit more complicated example is given by theory

\[ \mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 \]  

(12.80)

where \( \phi \) is a complex field with two components

\[ \begin{pmatrix} \text{Re}(\phi) \\ \text{Im}(\phi) \end{pmatrix} \text{ or } \begin{pmatrix} \text{Re}(\phi) + i \text{Im}(\phi) \\ \text{Re}(\phi) - i \text{Im}(\phi) \end{pmatrix} \equiv \begin{pmatrix} \phi \\ \phi^* \end{pmatrix}. \]  

(12.81)

Then the Lagrangian density and thus the action and equations of motion are invariant under

\[ \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} \to \begin{pmatrix} \phi' \\ \phi'^* \end{pmatrix} = \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} e^{i\alpha} = \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} + \alpha \begin{pmatrix} i\phi \\ -i\phi^* \end{pmatrix} \]  

(12.82)
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Then in terms of general transformations (12.64) this corresponds to

\[ \Delta \phi = \begin{pmatrix} i \phi \\ -i \phi^* \end{pmatrix} \]  

(12.83)

and from equation (12.73) we get

\[ j^\mu = i \left( (\partial_\mu \phi^*) \phi - (\partial_\mu \phi) \phi^* \right). \]  

(12.84)

If we couple this field to electrodynamic field

\[ \mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + j^\mu A_\mu \]  

(12.85)

then the conserved current corresponds to the electromagnetic current density.

Another important conserved current is the energy momentum tensor which describes conservations of the energy and momentum due to the symmetry of space and time translations. For a single real scalar field the energy momentum tensor is given by

\[ T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu - \delta^\mu_\nu, \]  

(12.86)

such that

\[ \partial_\mu T^\mu_\nu = 0. \]