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A note on asymptotic distribution of products of sums

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Abstract

The asymptotic behavior of product of the partial sums from a sequence of independent and identically distributed positive random variables have been studied by several papers. In Qi (Statist. Probab. Lett. 62 (2003) 93) the limit distribution for properly normalized products is proved when the underling distribution is in the domain of attraction of a stable law with exponent in (1,2]. In this note a similar result is obtained when the underlying distribution is in the domain of attraction of a stable law with exponent 1 and has a finite first moment.

MSC: 60F05

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1. Introduction and main result

Throughout this paper let $\{X, X_n,\ n\geqslant 1\}$ be a sequence of independent and identically distributed positive random variables and define the partial sum $S_n = \sum_{j=1}^n X_j$ for $n\geqslant 1$. Set $F(x) = P(X\leqslant x)$ for $x\geqslant 0$.

We are interested in asymptotic distribution of product $\prod_{j=1}^n S_j$. The study was initiated by Arnold and Villaseñor (1998) who obtained the limit distribution for the logarithm of the product $\prod_{j=1}^n S_j$ in case the underlying distribution of X is exponential when they studied the limit

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distribution for sums of records. Later on, Rempała and Wesołowski (2002) proved under the assumption $E(X^2) < \infty$ that

$$\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{\mathrm{d}} \mathrm{e}^{\sqrt{2}\mathscr{N}}, \tag{1.1}$$

where $\mathscr{N}$ is a standard normal random variable, $\mu = E(X)$ and $\gamma = \sigma/\mu$ with $\sigma^2 = \mathrm{Var}(X)$. Very recently, Qi (2003) extended (1.1) by assuming that the underlying distribution F is in the domain of attraction of a stable law with exponent $\alpha \in (1, 2]$.

Let L_α denote a stable law with exponent $\alpha \in (0, 2]$. Then L_α is determined, to within type, by a characteristic function of one of the following forms:

(ch1) $\phi(t) = \exp\{-t^2/2\}$ (normal case, $\alpha = 2$);
(ch2) $\phi(t) = \exp\{-|t|^\alpha(1 - \mathrm{i}\beta(\operatorname{sgn} t)\tan\frac{1}{2}\pi\alpha\}$ ($0 < \alpha < 1$ or $1 < \alpha < 2$, $-1 \leqslant \beta \leqslant 1$);
(ch3) $\phi(t) = \exp\{-|t|(1 + \mathrm{i}\beta(\operatorname{sgn} t)2/\pi \log|t|\}$ ($\alpha = 1$, $-1 \leqslant \beta \leqslant 1$).

See, e.g., Hall (1981), Bingham et al. (1987), or Qi (2003, Theorem 2.1). Recall that an underlying distribution F of X is said to be in the domain of attraction of a stable law L_α if there exists some constants $A_n > 0$ and $B_n \in R$ such that

$$\frac{S_n - B_n}{A_n} \xrightarrow{\mathrm{d}} \mathscr{L}_\alpha. \tag{1.2}$$

Since we assume in this paper that X is a positive random variable, the limit L_α in (1.2), to within type, will have a characteristic function of forms (ch1), (ch2) or (ch3) with the skewness parameter $\beta = 1$.

Define the generalized inverse of $1/(1 - F)$ by

$$U(x) = \inf\left\{t\colon \frac{1}{1 - F(t)} \geqslant x\right\}.$$

If the first moment of X exists, then set $\mu = E(X)$ and write

$$S(x) = E[(X - \mu)^2]I(|X - \mu| \leqslant x) \quad \text{for } x > 0.$$

Accordingly, denote the generalized inverse of $x^2/S(x)$ by $V(x)$

$$V(x) = \inf\left\{t\colon \frac{t^2}{S(t)} \geqslant x\right\}.$$

Set $A_n = U(n)$ if $\alpha < 2$, and $A_n = V(n)$ if $\alpha = 2$. Then one can always take $B_n = nE(XI(X \leqslant A_n))$ such that (1.2) holds for some stable law $\mathscr{L}_\alpha$ for any $\alpha \in (0, 2]$. Then from Loève (1977), the limit $\mathscr{L}_\alpha$ in (1.2) has a characteristic function as defined in (ch1), (ch2) or (ch3) with $\beta = 1$. If $\alpha \in (1, 2]$, one can choose $B_n = nE(X)$, then (1.2) holds with a limit $\mathscr{L}$ which is of same type as $\mathscr{L}_\alpha$.

With above choice for A_n, Qi (2003) showed that if $\alpha \in (1, 2]$

$$\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu^n}\right)^{\mu/A_n} \xrightarrow{\mathrm{d}} \mathrm{e}^{(\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}}, \tag{1.3}$$

where $\Gamma(\alpha + 1) = \int_0^\infty x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x$.

As mentioned in Qi (2003), an unsolved problem in case $\alpha \in (0,1]$ is whether there exist normalization constants C_n and D_n such that $\left(\prod_{j=1}^n S_j/D_n\right)^{C_n}$ converges in distribution to some non-degenerate distribution. This paper is devoted to the study of asymptotic distribution for the product of partial sums when the underlying distribution is in the domain of attraction of a stable law with exponent $\alpha = 1$ and $\mu = E(X) < \infty$.

As in Qi (2003), we define $A_n = U(n)$. But this time we need to set $B_n = nE(XI(X \leqslant A_n))$ such that (1.2) holds since (1.2) does not hold for any limit if one selects $B_n = n\mu$ in case $\alpha = 1$. From the proof in Qi (2003) we can reformulate (1.3) as

$$\left(\frac{\prod_{j=1}^n S_j}{n!\mu_n^n}\right)^{\mu/A_n} \xrightarrow{\mathrm{d}} \mathrm{e}^{(\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}_\alpha} \quad \text{for } \alpha \in (1,2], \tag{1.4}$$

where $\mu_n = E(XI(X \leqslant A_n))$.

Since X is positive, it is well-known from Loève (1977) that (1.2) is equivalent to the following condition

$$1 - F(x) = l(x)x^{-\alpha} \quad \text{for } x > 0, \tag{1.5}$$

where $l(x)$ is a slowly varying function at infinity.

Theorem 1. *Suppose* (1.5) *holds with* $\alpha = 1$ *and* $\mu = E(X) < \infty$. *Then* (1.4) *holds, i.e.,*

$$\left(\frac{\prod_{j=1}^n S_j}{n!\mu_n^n}\right)^{\mu/A_n} \xrightarrow{\mathrm{d}} \mathrm{e}^{\mathscr{L}_1}, \tag{1.6}$$

where $\mathscr{L}_1$ *is a stable distribution having a characteristic function defined in* (ch3) *with* $\beta = 1$.

We actually solve a conjecture proposed in Qi (2003). When $\alpha \in (0,1)$ (this implies $E(X) = \infty$) or $\alpha = 1$ but $E(X) = \infty$, the asymptotic distribution of the product of sums remains an unsolved problem. We guess some other probability tools and techniques may be required.

It is worth mentioning that Taylor's expansion plays an important role in both Qi (2003) and the present paper. For $\alpha \in (1,2]$, since $(S_j - j\mu)/j\mu$ converges to zero almost surely at a very fast rate, in Qi (2003) it is able to expand logarithm of the product of the partial sums into a weighted sum of random variables plus a sum of remainders. It turns out that the sum of the remainders is negligible. However, in this paper, $\alpha = 1$, this technique is no longer valid. The new techniques used in the proof of the paper include partial sum processes and new normalization constants $B_n = nE(XI(X \leqslant A_n))$. The proof also splits the logarithm of the product of the partial sums into two sums. A very delicate process is involved in selecting integers k_n so that all last $n - k_n$ terms can be expanded and their sum has a desired limit but the sum of the first k_n terms is negligible.

2. Proof

First we present two lemmas that will be used in the proof. For the proof of the first lemma see, e.g., Resnick (1986, Proposition 3.4 and its proof). The second lemma is an extension of Lemma 2.3 in Qi (2003) and its proof follows the same lines of the proof of the latter.

Lemma 1. *Assume condition* (1.5) *holds for some* $\alpha \in (0,2)$. *Then*

$$R_n(\cdot) := \frac{1}{A_n}(S_{[n\cdot]} - (\cdot)n\mu_n) \Rightarrow Y_\alpha(\cdot)$$

in $D[0,\infty)$, *where* $\{Y_\alpha(t),\ t \geqslant 0\}$ *is a Lévy process.*

Lemma 2. *Under the conditions of Theorem* 1

$$\sum_{j=1}^{n} \frac{\log(n+1/j)}{A_n}(X_j - \mu_n) \xrightarrow{\mathrm{d}} \mathscr{L}_1.$$

Now we are ready to prove Theorem 1. Note that

$$\begin{aligned}\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu_n^n}\right)^{\mu/A_n} &= \exp\left\{\frac{\mu}{A_n}\sum_{j=1}^{n}\log\frac{S_j}{j\mu_n}\right\}\\ &= \exp\left\{\frac{\mu}{A_n}\sum_{j=1}^{k_n}\log\frac{S_j}{j\mu} + \frac{\mu k_n}{A_n}\log\frac{\mu}{\mu_n} + \frac{\mu}{A_n}\sum_{j=k_n+1}^{n}\log\frac{S_j}{j\mu_n}\right\},\end{aligned}$$

where $\{k_n, n \geqslant 1\}$ is a sequence of integers with $1 \leqslant k_n < n$. If the sequence $\{k_n\}$ is selected in such a way that

$$\frac{k_n}{A_n}\log\frac{\mu}{\mu_n} \to 0, \tag{2.1}$$

$$\frac{\mu}{A_n}\sum_{j=1}^{k_n}\log\frac{S_j}{j\mu} \xrightarrow{\mathrm{p}} 0 \tag{2.2}$$

and

$$\frac{\mu}{A_n}\sum_{j=k_n+1}^{n}\log\frac{S_j}{j\mu_n} \xrightarrow{\mathrm{d}} \mathscr{L}_1, \tag{2.3}$$

then Theorem 1 is proved.

Since $1-F$ is regularly varying, we have $P(X > A_n) = 1 - F(A_n) \sim 1/n$ as $n \to \infty$. By Chebyshev's inequality, $A_n/n \sim A_n P(X > A_n) \leqslant E(XI(X > A_n)) =: \delta_n \to 0$ as $n \to \infty$. Another fact we will use is that $\log \mu/\mu_n \to 0$ since $\mu_n \to \mu$.

By the Kolmogorov's strong law of large numbers, with probability one, $S_j/j \to \mu$ as $j \to \infty$ from which we have $\log S_j/j\mu \to 0$ and $(\log(1+1/j))(S_j - j\mu) \to 0$. Therefore, with probability one,

$$W_n := \left|\frac{1}{n}\sum_{j=1}^{n}\log\frac{S_j}{j\mu}\right| + \left|\frac{1}{n}\sum_{j=1}^{n}\left(\log\left(1+\frac{1}{j}\right)\right)(S_j - j\mu)\right| \to 0,$$

which implies $W_n \to^{p} 0$. That is, for every $\varepsilon > 0$, $P(W_n > \varepsilon) \to 0$ as $n \to \infty$. It is easily seen that there exists a non-increasing sequence of positive numbers ε_n's, $\varepsilon_n \to 0$ such that

$$P(W_n > \varepsilon_n) \to 0. \tag{2.4}$$

Let $[x]$ denote the smallest integer larger than x. Set $i_n = [A_n]$ and $k_n = [A_n/\sqrt{\max(\varepsilon_{i_n}, \delta_n)}]$. Then $k_n/A_n \sim 1/\sqrt{\max(\varepsilon_{i_n}, \delta_n)} \to \infty$,

$$\begin{aligned} k_n \max(\varepsilon_{k_n}, \delta_n)/A_n &\sim \max(\varepsilon_{k_n}, \delta_n)/\sqrt{\max(\varepsilon_{i_n}, \delta_n)} \\ &\leqslant \sqrt{\max(\varepsilon_{k_n}, \delta_n)} \\ &\to 0 \end{aligned}$$

and

$$\begin{aligned} k_n/n &\sim A_n/(n\sqrt{\max(\varepsilon_{i_n}, \delta_n)}) \\ &\sim A_n P(X > A_n)/\sqrt{\max(\varepsilon_{i_n}, \delta_n)} \\ &\leqslant \delta_n/\sqrt{\max(\varepsilon_{i_n}, \delta_n)} \\ &\leqslant \sqrt{\delta_n} \to 0. \end{aligned}$$

Thus, we have proved

$$\frac{k_n}{A_n} \to \infty, \frac{k_n}{n} \to 0 \quad \text{and} \quad \frac{k_n \max(\varepsilon_{k_n}, \delta_n)}{A_n} \to 0. \tag{2.5}$$

From (2.4) we have

$$P(W_{k_n} > \varepsilon_{k_n}) \to 0,$$

i.e., $W_{k_n}/\varepsilon_{k_n} = O_p(1)$ (bounded in probability). Hence

$$\frac{k_n W_{k_n}}{A_n} = \frac{k_n \varepsilon_{k_n}}{A_n} \frac{W_{k_n}}{\varepsilon_{k_n}} \xrightarrow{p} 0.$$

This implies (2.2) and

$$\frac{1}{A_n} \sum_{j=1}^{k_n} \left(\log\left(1 + \frac{1}{j}\right)\right)(S_j - j\mu) \xrightarrow{p} 0. \tag{2.6}$$

It is easy to see that (2.1) follows from (2.5) since $\log \mu/\mu_n = \log(1 + \delta_n/\mu_n) \sim \delta_n/\mu_n \sim \delta_n$.

We still need to show (2.3). From Lemma 1 we conclude $Z_n := \sup_{0 \leqslant t \leqslant 1} |R_n(t)| \to^{d} \sup_{0 \leqslant t \leqslant 1} |Y_1(t)|$. Notice that

$$\frac{S_j}{j\mu_n} = 1 + \frac{S_j - j\mu_n}{j\mu_n} = 1 + \frac{A_n}{j\mu_n} R_n\left(\frac{j}{n}\right).$$

One can easily show that for $|x| \leqslant 1/2$

$$\log(1 + x) = x + \delta(x)x^2, \quad \text{where } |\delta(x)| \leqslant 2.$$

Since $\max_{k_n+1\leqslant j\leqslant n} |A_n/j\mu_n R_n(j/n)| \leqslant A_n/k_n\mu_n Z_n \to^{\mathrm{p}} 0$ as $n\to\infty$, we have for $k_{n+1}\leqslant j\leqslant n$

$$\log\frac{S_j}{j\mu_n} = \frac{A_n}{j\mu_n}R_n\left(\frac{j}{n}\right) + \left(\frac{A_n}{j\mu_n}R_n\left(\frac{j}{n}\right)\right)^2 d_j,$$

where $|d_j| = |\delta(A_n/j\mu_n R_n(j/n))|$ are uniformly dominated by 2 on the set $\{|A_n/k_n\mu_n Z_n|\leqslant 1/2\}$ which has a probability tending to 1 as $n\to\infty$. In order to show (2.3), it suffices to prove

$$\frac{1}{A_n}\sum_{j=k_n+1}^{n}\left(\frac{A_n}{j\mu_n}R_n\left(\frac{j}{n}\right)\right)^2 \xrightarrow{\mathrm{p}} 0 \tag{2.7}$$

and

$$\frac{\mu}{A_n}\sum_{j=k_n+1}^{n}\frac{A_n}{j\mu_n}R_n\left(\frac{j}{n}\right) \xrightarrow{\mathrm{d}} \mathscr{L}_1. \tag{2.8}$$

(2.7) follows from (2.5) by observing that

$$\frac{1}{A_n}\sum_{j=k_n+1}^{n}\left(\frac{A_n}{j\mu_n}R_n\left(\frac{j}{n}\right)\right)^2 = \frac{A_n}{\mu_n^2}\sum_{j=k_n+1}^{n}\frac{1}{j^2}\left(R_n\left(\frac{j}{n}\right)\right)^2 \leqslant \frac{A_n}{\mu_n^2}Z_n^2\sum_{j=k_n+1}^{\infty}j^2 \leqslant \frac{A_n}{k_n\mu_n^2}Z_n^2 \xrightarrow{\mathrm{p}} 0.$$

Let $c(j) = \log(1+1/j) - 1/j$. Then $|c(j)|\leqslant 1/j^2$. So we can rewrite the left-hand side of (2.8) as

$$\begin{aligned}
&\frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}\frac{1}{j}R_n\left(\frac{j}{n}\right)\\
&\quad= \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}\log\left(1+\frac{1}{j}\right)R_n\left(\frac{j}{n}\right) - \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}c(j)R_n\left(\frac{j}{n}\right)\\
&\quad= \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=k_n+1}^{n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu_n) - \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}c(j)R_n\left(\frac{j}{n}\right)\\
&\quad= \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu_n) - \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{k_n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu_n)\\
&\qquad - \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}c(j)R_n\left(\frac{j}{n}\right)\\
&\quad= \frac{\mu}{\mu_n}\sum_{j=1}^{n}\frac{\log\left(\frac{n+1}{j}\right)}{A_n}(X_j - \mu_n) - \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{k_n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu_n)\\
&\qquad - \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}c(j)R_n\left(\frac{j}{n}\right)\\
&\quad=: I_1 - I_2 - I_3.
\end{aligned}$$

From (2.5) and (2.6) we get

$$
\begin{aligned}
|I_2| &\leqslant \left|\frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{k_n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu)\right| + \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{k_n} j\log\left(1+\frac{1}{j}\right)(\mu-\mu_n)\\
&\leqslant \frac{\mu}{\mu_n}\left|\frac{1}{A_n}\sum_{j=1}^{k_n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu)\right| + \frac{\mu}{\mu_n}\frac{1}{A_n}\sum_{j=1}^{k_n} 2\delta_n\\
&\leqslant \frac{\mu}{\mu_n}\left|\frac{1}{A_n}\sum_{j=1}^{k_n}\log\left(1+\frac{1}{j}\right)(S_j - j\mu)\right| + \frac{\mu}{\mu_n}\frac{2k_n\delta_n}{A_n}\\
&\stackrel{\mathrm{p}}{\rightarrow} 0
\end{aligned}
$$

and

$$
|I_3| \leqslant \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}|c(j)|Z_n \leqslant \frac{\mu}{\mu_n}\sum_{j=k_n+1}^{n}\frac{Z_n}{j^2} \leqslant \frac{\mu}{\mu_n}\frac{Z_n}{k_n} \stackrel{\mathrm{p}}{\rightarrow} 0
$$

as $n \to \infty$. Finally, it follows from Lemma 2 that $I_1 \to^{\mathrm{d}} \mathscr{L}_1$. That proves (2.8).

The proof of Theorem 1 is complete.

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