On the Set of Limit Points of Normed Sums of Geometrically Weighted I.I.D. Bounded Random Variables

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For a sequence of nondegenerate i.i.d. bounded random variables \( \{Y_n, n \geq 1\} \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a constant \(b > 1\), it is shown for
\[
W_n = (b - 1) \sum_{i=1}^{n} b^{-i} Y_i \]
that
\[
S(F_Y) \subseteq \{c \in \mathbb{R} : \mathbb{P}(\{\omega \in \Omega : c \text{ is a limit point of } W_n(\omega)\}) = 1\}
= S(F_V) \subseteq [l, L]
\]
and that for almost every \(\omega \in \Omega\),
the set of limit points of \(W_n(\omega)\) coincides with the set \(S(F_V)\)

where \(S(F_Y)\) and \(S(F_V)\) are the spectrums of the distribution functions of \(Y_1\) and \(V = (b - 1) \sum_{i=1}^{\infty} b^{-i} Y_i\), respectively and \(l\) and \(L\) are the essential infimum of \(Y_1\) and the essential supremum of \(Y_1\), respectively. Examples are provided showing that, in general, the above two inclusions are proper.

Keywords Almost sure convergence; Essential infimum; Essential supremum; Geometric weights; Infinite Bernoulli convolution; Iterated logarithm type behavior; Law of the iterated logarithm; Limit points; Spectrum of a distribution function; Sums of geometrically weighted i.i.d. random variables.

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1. Introduction

At the origin of the current investigation is Rosalsky’s [16] study of iterated logarithm type behavior for geometrically weighted independent and identically distributed (i.i.d.) bounded random variables. The result of Rosalsky [16] pertaining to the current work will now be discussed. We recall that the essential infimum and the essential supremum of a bounded random variable \( Y \), denoted by \( \text{ess inf} Y \) and \( \text{ess sup} Y \), respectively, are defined by

\[
\text{ess inf} Y = \sup \{ y : \mathbb{P}(Y \geq y) = 1 \}
\]

and

\[
\text{ess sup} Y = \inf \{ y : \mathbb{P}(Y \leq y) = 1 \}.
\]

Rosalsky [16] obtained the following result, which describes the almost sure (a.s.) asymptotic fluctuation behavior of the partial sums \( \sum_{i=1}^{n} b^{i} Y_{i} \) where \( \{Y_{n}, n \geq 1\} \) is a sequence of nondegenerate i.i.d. bounded random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( b > 1 \) is a constant. The \( b^{i}, i \geq 1 \) are referred to as geometric weights and the partial sums \( \sum_{i=1}^{n} b^{i} Y_{i}, n \geq 1 \) are thus referred to as sums of geometrically weighted i.i.d. random variables.

**Theorem 1.1** (Rosalsky [16]). Let \( \{Y_{n}, n \geq 1\} \) be a sequence of nondegenerate i.i.d. bounded random variables and let \( b > 1 \). Then

\[
\limsup_{n \to \infty} \frac{\sum_{i=1}^{n} b^{i} Y_{i}}{b^{n+1}/(b - 1)} = L \quad \text{a.s.}
\]

and

\[
\liminf_{n \to \infty} \frac{\sum_{i=1}^{n} b^{i} Y_{i}}{b^{n+1}/(b - 1)} = l \quad \text{a.s.}
\]

where \( L = \text{ess sup} Y_{1} \) and \( l = \text{ess inf} Y_{1} \).

Theorem 1.1 sharpens a result of Teicher [21, Theorem 5], which asserts that the strong law of large numbers

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} b^{i} Y_{i}}{b^{n} c_{n}} = 0 \quad \text{a.s.}
\]

holds for every numerical sequence \( c_{n} \to \infty \) (no matter how slowly).

Now one of the most striking and fundamental results of probability theory is the renowned law of the iterated logarithm due to Hartman and Wintner [10] for a sequence of i.i.d. square integrable random variables. This result, which has long been celebrated as being a crowning achievement in the theory of sums of independent random variables, is stated as follows.

**Theorem 1.2** (Hartman and Wintner [10]). Let \( \{X_{n}, n \geq 1\} \) be a sequence of i.i.d. random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose that \( \mathbb{E}X_{1} = 0 \) and
\[ \mathbb{E}X_1^2 = 1. \] Then

\[ \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{\sqrt{2n \log \log n}} = 1 \quad \text{a.s. and} \quad \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{\sqrt{2n \log \log n}} = -1 \quad \text{a.s.} \]

Alternative proofs of the Hartman–Wintner theorem were discovered by Strassen [20], Heyde [11], Egorov [6], Teicher [21], Csörgő and Révész [4, p. 119], de Acosta [1], and Einmahl and Li [7].

We recall that for a given sequence \( \{t_n, n \geq 1\} \) in \( \mathbb{R} \), a point \( c \in [-\infty, \infty] \) said to be a limit point of \( \{t_n, n \geq 1\} \) if some subsequence of \( \{t_n, n \geq 1\} \) has \( c \) as its limit:

\[ \lim_{k \to \infty} t_{n_k} = c \]

for some positive integer sequence \( \{n_k, k \geq 1\} \)

with \( n_k < n_{k+1}, k \geq 1 \).

It is convenient to simply say that \( c \) is a limit point of \( t_n \).

**Remark 1.1.** (i) It is well known (see, e.g., Petrov [15, p. 248]) that the following sharpening of Theorem 1.2 holds: Under the hypotheses of Theorem 1.2,

\[ \left\{ c \in [-\infty, \infty] : \mathbb{P} \left( \left\{ \omega \in \Omega : c \text{ is a limit point of} \frac{\sum_{i=1}^{n} X_i(\omega)}{\sqrt{2n \log \log n}} \right\} = 1 \right\} = [-1, 1]. \] (1.1)

Hence, for all \( c \in [-1, 1] \), there exists an event \( \Omega_c \) with \( \mathbb{P} (\Omega_c) = 1 \) such that for all \( \omega \in \Omega_c \), there exists a strictly increasing sequence of positive integers \( \{n_k, k \geq 1\} \) such that

\[ \lim_{k \to \infty} \frac{\sum_{i=1}^{n_k} X_i(\omega)}{\sqrt{2n_k \log \log n_k}} = c. \]

The integer sequence \( \{n_k, k \geq 1\} \) is of course random; that is, the \( n_k, k \geq 1 \) depend on \( \omega \in \Omega_c \).

(ii) An even stronger version of the limit point result (1.1) indeed prevails; namely (see, e.g., Stout [19, p. 294]),

\[ \text{the set of limit points of} \frac{\sum_{i=1}^{n} X_i(\omega)}{\sqrt{2n \log \log n}} \text{ coincides} \]

with the set \([-1, 1]\) for almost every \( \omega \in \Omega \).

(1.2)

The difference between (1.1) and (1.2) is that for \( c \in [-1, 1] \), (1.1) does not preclude the event of probability 1

\[ \left\{ \omega \in \Omega : c \text{ is a limit point of} \frac{\sum_{i=1}^{n} X_i(\omega)}{\sqrt{2n \log \log n}} \right\} \]
from depending on $c$ whereas (1.2) ensures that there exists an event $\Omega^*$ with $\mathbb{P}(\Omega^*) = 1$ such that for all $c \in [-1, 1]$ and all $\omega \in \Omega^*$,

$$c \text{ is a limit point of } \frac{\sum_{i=1}^{n} X_i(\omega)}{2n \log \log n};$$

that is, $\Omega^*$ does not depend on $c \in [-1, 1]$.

In the current work, for a sequence of nondegenerate i.i.d. bounded random variables $\{Y_n, n \geq 1\}$ and a constant $b > 1$, we investigate which points $c \in \mathbb{R}$ are a.s. limit points of the sequence of normed weighted sums $\{W_n, n \geq 1\}$ defined by

$$W_n = \frac{\sum_{i=1}^{n} b^i Y_i}{b^n + 1/(b - 1)}, \quad n \geq 1$$

(1.3)

whose a.s. lim sup and a.s. lim inf were identified in Theorem 1.1.

Let

$$V_n = (b - 1) \sum_{i=1}^{n} b^{-i} Y_i, \quad n \geq 1.$$ (1.4)

Then

$$(b - 1) \sum_{i=1}^{\infty} \mathbb{E}(b^{-i} Y_i) = (b - 1) (\mathbb{E} Y_1) \sum_{i=1}^{\infty} b^{-i} = \mathbb{E} Y_1$$

and

$$(b - 1)^2 \sum_{i=1}^{\infty} \text{Var}(b^{-i} Y_i) = (b - 1)^2 (\text{Var} Y_1) \sum_{i=1}^{\infty} b^{-2i} = \frac{b - 1}{b + 1} \text{Var} Y_1$$

and so by the Khintchine–Kolmogorov convergence theorem (see, e.g., Chow and Teicher [3, p. 113]),

$$V_n \text{ converges a.s. to a random variable } V$$

(1.5)

with

$$\mathbb{E} V = \mathbb{E} Y_1 \text{ and } \text{Var} V = \frac{b - 1}{b + 1} \text{Var} Y_1.$$ (1.6)

A pertinent observation is that

$$W_n \text{ and } V_n \text{ are identically distributed, } n \geq 1.$$ (1.7)

For a random variable $X$ with distribution function $F_X(x) = \mathbb{P}(X \leq x), x \in \mathbb{R}$, we let $S(F_X)$ denote the spectrum of $F_X$; that is

$$S(F_X) = \{c \in \mathbb{R} : F_X(c - e) < F_X(c + e) \text{ for all } e > 0\}.$$
In Theorem 3.1, it is shown inter alia that
\[ \{c \in \mathbb{R} : \mathbb{P}(\{\omega \in \Omega : c \text{ is a limit point of } W_n(\omega)\}) = 1\} = S(F_v) \]
and in Theorem 3.2, it is shown that for almost every \( \omega \in \Omega \),
the set of limit points of \( W_n(\omega) \) coincides with the set \( S(F_v) \).

Consequently, Theorem 3.2 bears the same relation to Theorem 3.1 for sums of geometrically weighted i.i.d. random variables as in Theorem 1.1 as does (1.2) to (1.1) for sums of i.i.d. random variables as in Theorem 1.2.

The plan of this article is as follows. In Section 2, we prove two preliminary lemmas. The main results of this article, Theorems 3.1 and 3.2, are stated and proved in Section 3. In Section 4, two interesting examples are presented.

2. Preliminary Lemmas

To establish the main results, two lemmas are needed. The first lemma concerns a numerical sequence and is used to prove the second lemma which concerns an arbitrary sequence of random variables.

**Lemma 2.1.** Let \( \{t_n, n \geq 1\} \) be a sequence in \( \mathbb{R} \) and let \( c \in \mathbb{R} \). Then
\[ c \text{ is a limit point of } \{t_n, n \geq 1\} \tag{2.1} \]
if and only if
\[ \text{for all } \varepsilon > 0, \quad |t_n - c| < \varepsilon \text{ for infinitely many } n. \tag{2.2} \]

**Proof.** *Necessity.* This implication is evident.

*Sufficiency.* Suppose that (2.2) holds. Choose \( n_1 \geq 1 \) such that \( |t_{n_1} - c| < 1 \) and \( n_2 > n_1 \) such that \( |t_{n_2} - c| < \frac{1}{2} \). Suppose, inductively, that for some \( k \geq 2 \), there exists integers \( n_1, \ldots, n_k \) with \( 1 \leq n_1 < \cdots < n_k \) such that
\[ |t_{n_j} - c| < \frac{1}{j} \quad \text{for } j = 1, \ldots, k. \]

Choose \( n_{k+1} > n_k \) so that
\[ |t_{n_{k+1}} - c| < \frac{1}{k+1}. \]
Thus, \( \{n_k, k \geq 1\} \) is a strictly increasing sequence of positive integers with
\[ |t_{n_k} - c| < \frac{1}{k}, \quad k \geq 1. \]

Let \( \varepsilon > 0 \) and let \( K \) be such that \( K \geq 1/\varepsilon \). Then for all \( k \geq K \),
\[ |t_{n_k} - c| < \frac{1}{k} \leq \frac{1}{K} \leq \varepsilon \]
and so \( \lim_{k \to \infty} t_{n_k} = c \); that is, (2.1) holds. \( \square \)
Lemma 2.2. Let \( \{T_n, n \geq 1\} \) be a sequence of random variables and let \( c \in \mathbb{R} \). Then the following three statements are equivalent:

(i) \( c \) is a limit point of \( T_n \) a.s.
(ii) For all \( \epsilon > 0 \), \( \mathbb{P}(T_n \in (c - \epsilon, c + \epsilon) \text{ i.o. (} n)) = 1 \).
(iii) \( \mathbb{P}\left(\bigcap_{j=1}^{\infty}\left[T_n \in \left(c - \frac{1}{j}, c + \frac{1}{j}\right) \text{ i.o. (} n\right]\right) = 1 \).

Proof. The equivalence between (ii) and (iii) is evident and the equivalence between (i) and (iii) follows immediately from Lemma 2.1. \( \square \)

Remark 2.1. Stout [19, p. 332] defines a point \( c \in \mathbb{R} \) to be a recurrent state of the sequence \( \{T_n, n \geq 1\} \) if (ii) of Lemma 2.2 holds.

3. The Main Results

With the preliminaries accounted for, the main results may be stated and proved. Throughout the rest of the article, let \( \{W_n, n \geq 1\}, \{V_n, n \geq 1\}, \) and \( V \) be as in (1.3)–(1.5), respectively, let \( F_{Y_1} \) and \( F_V \) denote the distribution functions of \( Y_1 \) and \( V \), respectively, let

\[ C = \{c \in \mathbb{R} : \mathbb{P}(\{\omega \in \Omega : c \text{ is a limit point of } W_n(\omega)\}) = 1\}, \]

and let

\[ C_{(\omega)} \text{ denote the set of limit points of } W_n(\omega), \ \omega \in \Omega. \]

Theorem 3.1. Let \( \{Y_n, n \geq 1\} \) be a sequence of nondegenerate i.i.d. bounded random variables defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( b > 1 \). Then

\[ S(F_{Y_1}) \subseteq C = S(F_V) \subseteq [l, L] \] (3.1)

where \( l = \text{ess inf } Y_1 \) and \( L = \text{ess sup } Y_1 \).

Proof. To prove \( S(F_{Y_1}) \subseteq C \), let \( c \in S(F_{Y_1}) \). Set

\[ p_\epsilon = F_{Y_1}\left(c + \frac{\epsilon}{2}\right) - F_{Y_1}\left(c - \frac{\epsilon}{2}\right) = \mathbb{P}\left(Y_1 \in \left(c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}\right)\right), \ \epsilon > 0. \]

Then \( p_\epsilon > 0 \) for all \( \epsilon > 0 \) (since \( c \in S(F_{Y_1}) \)) and \( p_\epsilon < 1 \) for all sufficiently small \( \epsilon > 0 \) (since \( Y_1 \) is nondegenerate). Let

\[ n_k = k^2 \quad \text{and} \quad j_k = n_k + 1 - \left[\frac{\log k}{\log p_\epsilon^{-1}}\right], \ k \geq 1 \]

where \( \epsilon > 0 \) is small enough so that \( p_\epsilon < 1 \). Note that

\[ n_k + 1 - j_k = \left[\frac{\log k}{\log p_\epsilon^{-1}}\right] \to \infty \quad \text{as } k \to \infty. \] (3.2)
Now for \( k \geq 1, \)
\[
n_k + 1 - j_k \leq \frac{\log k}{\log p_e^{-1}} = \frac{\log k^{-1}}{\log p_e}
\]
or, equivalently,
\[
p_e^{n_k + 1 - j_k} \geq k^{-1}. \tag{3.3}
\]
Moreover,
\[
j_k < n_k < j_{k+1} \quad \text{for all large } k. \tag{3.4}
\]

Then, since the \( \{Y_n, n \geq 1\} \) are independent,
\[
\sum_{k=1}^{\infty} \mathbb{P}\left( \bigcap_{i=j_k}^{n_k} \left[ Y_i \in \left( c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2} \right) \right] \right) = \sum_{k=1}^{\infty} \left( F_{Y_1} \left( c + \frac{\varepsilon}{2} \right) - F_{Y_1} \left( c - \frac{\varepsilon}{2} \right) \right)^{n_k - j_k + 1}
\]
\[
= \sum_{k=1}^{\infty} p_e^{n_k - j_k + 1} \geq \sum_{k=1}^{\infty} k^{-1} \quad \text{(by (3.3))} = \infty.
\]
Moreover, the independence of the \( \{Y_n, n \geq 1\} \) and (3.4) ensure that for all large \( K, \)
\[
\bigcap_{i=1}^{n_k} \left[ Y_i \in \left( c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2} \right) \right], \quad k \geq K \text{ are independent}
\]
and so by the Borel–Cantelli lemma,
\[
\mathbb{P}\left( \bigcap_{i=j_k}^{n_k} \left[ Y_i \in \left( c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2} \right) \right] \right) \quad \text{i.o. (k)} = 1. \tag{3.5}
\]

Next, note that
\[
\left| \sum_{i=1}^{j_k-1} b^i Y_i \right| \leq \sum_{i=1}^{j_k-1} b^i |Y_i| \leq \sum_{i=1}^{j_k-1} b^i (|L| \vee |l|) \leq \frac{b^{j_k}}{b - 1} (|L| \vee |l|)
\]
and so
\[
\frac{\left| \sum_{i=1}^{j_k-1} b^i Y_i \right|}{b^{n_k+1}/(b - 1)} \leq \frac{|L| \vee |l|}{b^{n_k+1-j_k}} \to 0 \quad \text{a.s. as } k \to \infty \quad \text{(by (3.2)).} \tag{3.6}
\]
Moreover, it follows from (3.5) that with probability 1, for infinitely many \( k, \)
\[
\sum_{i=j_k}^{n_k} b^i Y_i \\ b^{n_k+1}/(b - 1) \leq \sum_{i=j_k}^{n_k} b^i (c \pm \varepsilon) \\ \geq \frac{b^{n_k+1}}{b^{n_k+1-j_k}} \left( c \pm \frac{\varepsilon}{2} \right).
\]
\[= \left(1 - \frac{1}{b^{n+1}}\right) \left(c \pm \frac{\varepsilon}{2}\right)\]
\[
\begin{cases}
\leq c \pm \frac{2}{3}\varepsilon \\
\geq c \pm \frac{2}{3}\varepsilon
\end{cases}
\]
(3.7)

provided \(k\) is sufficiently large (by (3.2)). Then by (3.6) and (3.7), with probability 1, for infinitely many \(k\),

\[W_{n_k} = \sum_{i=1}^{b^{n_k}+1/(b-1)} b^i Y_i + \sum_{i=b^{n_k}}^{n_k} b^i Y_i \in (c - \varepsilon, c + \varepsilon);\]

that is,

\[\mathbb{P}\left(W_{n_k} \in (c - \varepsilon, c + \varepsilon) \text{ i.o. (k)}\right) = 1\]

and, a fortiori,

\[\mathbb{P}\left(W_n \in (c - \varepsilon, c + \varepsilon) \text{ i.o. (n)}\right) = 1.\]

Then by Lemma 2.2,

\[\mathbb{P}\left(\{\omega \in \Omega : c \text{ is a limit point of } W_n(\omega)\}\right) = 1\]

thereby proving that \(S(F_Y) \subseteq C\).

Next, it will be shown that

\[S(F_Y) \subseteq C.\] (3.9)

Let \(c \in S(F_Y)\) and let \(\varepsilon > 0\) be arbitrary. Let

\[q_c = F_Y\left(c + \frac{\varepsilon}{4}\right) - F_Y\left(c - \frac{\varepsilon}{4}\right) = \mathbb{P}\left(c - \frac{\varepsilon}{4} < V \leq c + \frac{\varepsilon}{4}\right).\]

Then \(q_c > 0\) since \(c \in S(F_Y)\). Note that

\[|V - V_n| \leq (b - 1) \sum_{i=n+1}^{\infty} b^{-i} |Y_i| \leq \frac{|L| \vee |l|}{b^n} \text{ a.s.}, n \geq 1.\] (3.10)

Let \(N(\varepsilon)\) be a positive integer such that

\[\frac{|L| \vee |l|}{b^{N(\varepsilon)}} < \frac{\varepsilon}{4}\] (3.11)

Set

\[U(m, n) = \frac{b - 1}{b^{n+1}} \sum_{i=m+1}^{n} b^i Y_i, \quad n > m \geq 1.\]
Then for \(n > m \geq 1\),
\[
U(m, n) - W_n = \frac{b - 1}{bn+1} \left( \sum_{i=m+1}^{n} b^i Y_i - \sum_{i=1}^{n} b^i Y_i \right) = -\frac{b - 1}{bn+1} \left( \sum_{i=1}^{m} b^i Y_i \right) \\
= -\frac{b - 1}{bn-m} \left( \frac{\sum_{i=m+1}^{n} b^i Y_i}{b^{n+1}} \right) = -\frac{W_m}{b^{n-m}}. \tag{3.12}
\]

Now for \(j \geq 1\),
\[
W_j \leq \frac{(b - 1) \left( \sum_{i=1}^{j} b^i \right)}{b^{j+1}} \leq \frac{b - 1}{b^{j+1}} \cdot \frac{b^{j+1} - b}{b - 1} \cdot L \leq |L| \text{ a.s.}
\]
and, similarly,
\[
W_j \geq -|l| \text{ a.s.}
\]

Thus,
\[
|W_j| \leq |L| \lor |l| \text{ a.s., } j \geq 1. \tag{3.13}
\]

Then, by (3.12), (3.13), and (3.11), whenever \(m \geq 1\) and \(n \geq m + N(\varepsilon)\) we have
\[
|U(m, n) - W_n| = \frac{|W_n|}{b^{n-m}} \leq \frac{|L| \lor |l|}{b^{n-m}} < \frac{\varepsilon}{4} \text{ a.s.} \tag{3.14}
\]
and
\[
\frac{|W_m|}{b^{n-m}} \leq \frac{|L| \lor |l|}{b^{n-m}} < \frac{\varepsilon}{4} \text{ a.s.} \tag{3.15}
\]

Now, whenever \(m \geq 1\) and \(n \geq m + N(\varepsilon)\),
\[
0 < q_\varepsilon = \mathbb{P} \left( c - \varepsilon < U \leq c + \frac{\varepsilon}{4} \right) \\
= \mathbb{P} \left( c - \varepsilon < U \leq c + \frac{\varepsilon}{4} \land \frac{-\varepsilon}{4} < V_n - V < \frac{\varepsilon}{4} \right) \text{ (by (3.10) and (3.11))} \\
\leq \mathbb{P} \left( c - \varepsilon < V_n < c + \frac{\varepsilon}{2} \right) \\
= \mathbb{P} \left( c - \frac{\varepsilon}{2} < W_n < c + \frac{\varepsilon}{2} \right) \text{ (by (1.7))} \\
= \mathbb{P} \left( c - \frac{\varepsilon}{2} < W_n < c + \frac{\varepsilon}{2} \land \frac{-\varepsilon}{4} < U(m, n) - W_n < \frac{\varepsilon}{4} \right) \text{ (by (3.14))} \\
\leq \mathbb{P} \left( c - \frac{3}{4} \varepsilon < U(m, n) < c + \frac{3}{4} \varepsilon \right).
\]

Let \(n_k = kN(\varepsilon)\), \(k \geq 1\). Then
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( c - \frac{3}{4} \varepsilon < U(n_k, n_{k+1}) < c + \frac{3}{4} \varepsilon \right) = \infty
\]
and since \( \{U(n_k, n_{k+1}) \, : \, k \geq 1 \} \) is a sequence of independent random variables, we have by the Borel–Cantelli lemma that

\[
P(c - \frac{3}{4} \varepsilon < U(n_k, n_{k+1}) < c + \frac{3}{4} \varepsilon \text{ i.o. (} k \text{) }) = 1. \tag{3.16}
\]

Now, whenever \( m \geq 1 \) and \( n \geq m + N(\varepsilon) \), we have up to an event of probability 0 that

\[
\begin{align*}
&\left[ c - \frac{3}{4} \varepsilon < U(m, n) < c + \frac{3}{4} \varepsilon \right] \\
= &\left[ c - \frac{3}{4} \varepsilon < W_n - \frac{W_m}{b^{n-m}} < c + \frac{3}{4} \varepsilon \right] \quad \text{(by (3.12))} \\
= &\left[ c - \frac{3}{4} \varepsilon < W_n - \frac{W_m}{b^{n-m}} < c + \frac{3}{4} \varepsilon, \frac{\varepsilon}{4} < \frac{W_m}{b^{n-m}} < \frac{\varepsilon}{4} \right] \quad \text{(by (3.15))} \\
\subseteq &\left[ c - \varepsilon < W_n < c + \varepsilon \right].
\end{align*}
\]

Then, recalling (3.16),

\[
1 = P\left( c - \frac{3}{4} \varepsilon < U(n_k, n_{k+1}) < c + \frac{3}{4} \varepsilon \text{ i.o. (} k \text{) } \right) \\
\leq P\left( c - \varepsilon < W_{n+1} < c + \varepsilon \text{ i.o. (} k \text{) } \right)
\]

and, a fortiori,

\[
P(c - \varepsilon < W_n < c + \varepsilon \text{ i.o. (} n \text{) } ) = 1.
\]

Thus, \( c \in \mathcal{C} \) by Lemma 2.2 since \( \varepsilon > 0 \) is arbitrary thereby establishing (3.9).

It will now be shown that

\[
\mathcal{C} \subseteq S(F_V). \tag{3.17}
\]

Suppose that \( c \notin S(F_V) \). Then for some \( \varepsilon_0 > 0 \),

\[
F_V(c + 2\varepsilon_0) - F_V(c - 2\varepsilon_0) = 0. \tag{3.18}
\]

Now recalling (3.10), we have for all large \( n \) that

\[
|V - V_n| < \varepsilon_0 \quad \text{a.s.} \tag{3.19}
\]

Then for all large \( n \),

\[
P(c - \varepsilon_0 < W_n < c + \varepsilon_0) \\
= P(c - \varepsilon_0 < V_n < c + \varepsilon_0) \quad \text{(by (1.7))} \\
= P(c - \varepsilon_0 < V_n < c + \varepsilon_0, -\varepsilon_0 < V - V_n < \varepsilon_0) \quad \text{(by (3.19))} \\
\leq P(c - 2\varepsilon_0 < V < c + 2\varepsilon_0) \\
\leq F_V(c + 2\varepsilon_0) - F_V(c - 2\varepsilon_0) = 0 \quad \text{(by (3.18))}
\]
and so

\[ \text{IP} \left( c - \varepsilon_0 < W_n < c + \varepsilon_0 \text{ i.o. } (n) \right) = 0. \]

Hence, by Lemma 2.2, \( c \notin C \). Consequently, (3.17) holds and so, recalling (3.9), \( C = S(F_v) \).

It remains to prove that \( C \subseteq [l, L] \). Let \( c \in C \); that is, \( c \in \mathbb{R} \) satisfies (3.8). Then by Theorem 1.1,

\[ l = \liminf_{n \to \infty} W_n \leq c \leq \limsup_{n \to \infty} W_n \leq L \quad \text{a.s.} \]

and so \( c \in [l, L] \) thereby proving that \( C \subseteq [l, L] \). (Alternatively, it can be noted that \( V = (b - 1) \sum_{i=1}^{\infty} b^{-i} Y_i \in [l, L] \) a.s. and so \( C = S(F_v) \subseteq [l, L] \).) \( \square \)

**Remark 3.1.** (i) Recalling (1.5) and (1.6),

\[ V_n \to V \quad \text{a.s. and } V \text{ is nondegenerate} \quad (3.20) \]

and recalling Theorem 1.1,

\[ \liminf_{n \to \infty} W_n = l \quad \text{a.s. and } \limsup_{n \to \infty} W_n = L \quad \text{a.s.} \quad (3.21) \]

Thus, in view of (3.20) and (3.21), we see that \( \{V_n, n \geq 1\} \) and \( \{W_n, n \geq 1\} \) exhibit entirely different asymptotic behavior although their marginal distributions coincide for all \( n \geq 1 \) as was noted in (1.7).

(ii) It follows from (3.20) that \( V_n \overset{d}{\to} V \) and hence by (1.7)

\[ W_n \overset{d}{\to} V. \quad (3.22) \]

It is interesting to contrast (3.21) and (3.22) with what prevails for the sequence

\[ H_n \equiv \frac{\sum_{i=1}^{n} X_i}{\sqrt{2n \log \log n}}, \quad n \geq 3 \]

in the Hartman–Wintner Theorem (Theorem 1.2 above) where \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. random variables with \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}X_1^2 = 1 \). Recall that Theorem 1.2 asserts that

\[ \liminf_{n \to \infty} H_n = -1 \quad \text{a.s. and } \limsup_{n \to \infty} H_n = 1 \quad \text{a.s.} \quad (3.23) \]

Using Chebyshev’s inequality, \( H_n \overset{p}{\to} 0 \) which is equivalent to

\[ H_n \overset{d}{\to} 0. \quad (3.24) \]

While (3.21) and (3.23) are analogous results in terms of their structure, (3.22) and (3.24) are structurally different since \( V \) is nondegenerate whereas the random variable 0 is degenerate.
(iii) We also observe that there does not exist a random variable $W$ such that $W_n \overset{p}{\to} W$. For if such a random variable $W$ exists, then $W_n \overset{d}{\to} W$ whence by (3.22) necessarily $W$ and $V$ are identically distributed and, moreover, there exists a (nonrandom) subsequence $\{n_k, k \geq 1\}$ such that $W_{n_k} \to W$ a.s. But by the Kolmogorov 0–1 law, $W = \limsup_{k \to \infty} W_{n_k}$ is degenerate which contradicts $V$ being nondegenerate.

In the next theorem, we show that for almost every $\omega \in \Omega$, the (random) set $\mathcal{C}_{(\omega)}$ equals the (nonrandom) set $S(F_V)$ (see Theorem 3.1). This is a strengthening of the assertion “$c \in S(F_V)$ if and only if $c \in \mathbb{R}$ and with probability 1, $c$ is a limit point of $W_n$” obtained from Theorem 3.1. Theorem 3.1 is used to prove Theorem 3.2.

**Theorem 3.2.** Let $\{Y_n, n \geq 1\}$ be a sequence of nondegenerate i.i.d. bounded random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $b > 1$. Then

$$\mathcal{C}_{(\omega)} = S(F_V) \quad \text{for almost every } \omega \in \Omega. \quad (3.25)$$

**Proof.** We first prove that

$$S(F_V) \subseteq \mathcal{C}_{(\omega)} \quad \text{for almost every } \omega \in \Omega. \quad (3.26)$$

For each $x \in S(F_V)$, by Theorem 3.1 there exists an event $\Omega_x$ with $\mathbb{P}(\Omega_x) = 1$ such that for all $\omega \in \Omega_x$,

$$x \text{ is a limit point of } W_n(\omega).$$

Since $S(F_V)$ is closed and since $S(F_V) \subseteq [l, L]$ by Theorem 3.1, $S(F_V)$ is compact. Thus, (see, e.g., Royden [17, p. 163]) $S(F_V)$ is separable; that is, $S(F_V)$ contains a countable set $K$ which is dense in $S(F_V)$. Set $\Omega^{(0)} = \bigcap_{x \in K} \Omega_x$. Since $K$ is countable, $\Omega^{(0)}$ is an event with $\mathbb{P}(\Omega^{(0)}) = 1$. Now for every $\omega \in \Omega^{(0)}$ and every $x \in K$,

$$x \text{ is a limit point of } W_n(\omega). \quad (3.27)$$

Fix $\omega_0 \in \Omega^{(0)}$ and $x_0 \in S(F_V)$. Let $\varepsilon > 0$. Since $K$ is dense in $S(F_V)$, there exists $x_1 \in K$ such that $|x_0 - x_1| < \varepsilon/2$. Now by (3.27) and Lemma 2.1,

$$|W_n(\omega_0) - x_1| < \frac{\varepsilon}{2} \quad \text{for infinitely many } n.$$

Then

$$|W_n(\omega_0) - x_0| \leq |W_n(\omega_0) - x_1| + |x_1 - x_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for infinitely many $n$

and so by Lemma 2.1, $x_0 \in \mathcal{C}_{(\omega_0)}$. Since $\omega_0 \in \Omega^{(0)}$ and $x_0 \in S(F_V)$ are arbitrary and $\mathbb{P}(\Omega^{(0)}) = 1$, the assertion (3.26) holds.

Next, it will be shown that

$$\mathcal{C}_{(\omega)} \subseteq S(F_V) \quad \text{for almost every } \omega \in \Omega. \quad (3.28)$$
Set
\[
\Omega^{(1)} = \left\{ \omega \in \Omega : \liminf_{n \to \infty} W_n(\omega) = l \text{ and } \limsup_{n \to \infty} W_n(\omega) = L \right\}.
\]

Then \( \mathbb{P}(\Omega^{(1)}) = 1 \) by Theorem 1.1 and
\[
\Omega^{(1)} \subseteq \left\{ \omega \in \Omega : \mathcal{C}_{(\omega)} \subseteq [l, L] \right\}. \tag{3.29}
\]

Now \( S(F_Y) \subseteq [l, L] \) by Theorem 3.1. If \( S(F_Y) = [l, L] \), then by (3.29)
\[
\Omega^{(1)} \subseteq \left\{ \omega \in \Omega : \mathcal{C}_{(\omega)} \subseteq S(F_Y) \right\}
\]
and so (3.28) holds since \( \mathbb{P}(\Omega^{(1)}) = 1 \). Thus, it will be assumed that \( S(F_Y) \) is a proper subset of \([l, L]\). Now by Theorem 3.1,
\[
[l, L] \subseteq S(F_Y) \subseteq S(F_Y) \subseteq [l, L]
\]
and since \( S(F_Y) \) is a closed set, it follows that
\[
[l, L] - S(F_Y) = [l, L] \cap (S(F_Y))^c = (l, L) \cap (S(F_Y))^c
\]
is an open set and hence (see, e.g., Royden [17, p. 39]) can be expressed as
\[
[l, L] - S(F_Y) = \bigcup_{i=1}^{\infty} (a_i, b_i) \tag{3.30}
\]
for some collection \( \{(a_i, b_i), i \geq 1\} \) of (nonempty) open intervals. Suppose we can show that there exists an event \( \Omega^{(2)} \subseteq \Omega^{(1)} \) with \( \mathbb{P}(\Omega^{(2)}) = 1 \) such that
\[
[l, L] - S(F_Y) \subseteq [l, L] - \mathcal{C}_{(\omega)}, \quad \omega \in \Omega^{(2)}. \tag{3.31}
\]

Then
\[
\mathcal{C}_{(\omega)} \subseteq S(F_Y), \quad \omega \in \Omega^{(2)}
\]
which establishes (3.28). We now verify (3.31) for some event \( \Omega^{(2)} \subseteq \Omega^{(1)} \) with \( \mathbb{P}(\Omega^{(2)}) = 1 \). Note that for all \( k \geq 1 \), it follows from (3.10) that there exists an integer \( N_k \geq 1 \) such that
\[
|V - V_n| \leq \frac{1}{k} \text{ a.s. for all } n \geq N_k. \tag{3.32}
\]

Then, for all \( i \geq 1, k \geq 1, \) and \( n \geq N_k \),
\[
\mathbb{P}\left(V_n \in \left(a_i + \frac{1}{k}, b_i - \frac{1}{k}\right)\right)
= \mathbb{P}\left(V_n \in \left(a_i + \frac{1}{k}, b_i - \frac{1}{k}\right), |V - V_n| \leq \frac{1}{k}\right) \quad \text{(by (3.32))}
\]
\[
\leq \mathbb{P} \left( a_i < V < b_i \right) = 0 \text{ (since } (a_i, b_i) \subseteq \left[ l, L \right] - S(F_V) \text{)}
\]
and, hence, by (1.7), for all \( i \geq 1, k \geq 1, \) and \( n \geq N_k \)
\[
\mathbb{P} \left( W_n \in \left( a_i + \frac{1}{k}, b_i - \frac{1}{k} \right) \right) = 0. \tag{3.33}
\]

Set
\[
\Omega^{(2)} = \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \left( \Omega^{(1)} \cap \bigcap_{n=N_k}^{\infty} \left[ W_n \not\in \left( a_i + \frac{1}{k}, b_i - \frac{1}{k} \right) \right] \right).
\]
Then, \( \Omega^{(2)} \subseteq \Omega^{(1)} \) and it follows from (3.33) that \( \mathbb{P}(\Omega^{(2)}) = 1. \) Now
\[
\Omega^{(2)} \subseteq \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \left( \bigcap_{n=N_k}^{\infty} \left[ W_n \not\in \left( a_i + \frac{1}{k}, b_i - \frac{1}{k} \right) \right] \right)
= \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \left( \Omega^{(1)} \cap \bigcap_{n=N_k}^{\infty} \left[ W_n \not\in \left( a_i + \frac{1}{k}, b_i - \frac{1}{k} \right) \right] \right)
= \bigcap_{i=1}^{\infty} \left( \Omega^{(1)} \cap \left\{ \omega \in \Omega : (a_i, b_i) \subseteq \left[ l, L \right] - C_{(\omega)} \right\} \right)
= \Omega^{(1)} \cap \left\{ \omega \in \Omega : [l, L] - S(F_V) \subseteq \left[ l, L \right] - C_{(\omega)} \right\} \quad \text{(by (3.30))}
\]
thereby proving (3.31) and, hence, (3.28) as was noted above. Combining (3.26) and (3.28) yields the conclusion (3.25). \( \square \)

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

**Corollary 3.1.** Let \( \{Y_n, n \geq 1\} \) be a sequence of nondegenerate i.i.d. bounded random variables defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( b > 1. \) If \( S(F_{Y_1}) = [l, L] \) where \( l = \text{ess inf } Y_1 \) and \( L = \text{ess sup } Y_1, \) then
\[
C_{(\omega)} = C = [l, L] \text{ for almost every } \omega \in \Omega.
\]

### 4. Some Interesting Examples

In this section, we present two interesting examples pertaining to Theorem 3.1. The first example shows that, in general, the first inclusion in (3.1) of Theorem 3.1 is proper. In this example, \( S(F_{Y_1}) = \{-1, 1\} \) but \( C = [-1, 1], \)

**Example 4.1.** Let \( \{Y_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}. \) Then \( S(F_{Y_1}) = \{-1, 1\}. \) In Theorem 3.1, take \( b = 2. \) Now
\[
V_n = \sum_{i=1}^{n} 2^{-i} Y_i \text{ converges a.s. to a random variable } V.
\]
It is well known that \( V \) is uniformly distributed on \([-1, 1]\) (see, e.g., Chow and Teicher [3, Exercise 8.3.7, p. 293]) for a routine characteristic function argument). Thus, \( \mathbb{C} = [-1, 1] \) by Theorem 3.1. The earliest reference we can find to \( V \) having a uniform distribution is Jessen and Wintner [12].

**Remark 4.1.** One of the great problems of twentieth century probability theory which has not yet been completely solved is that of identifying the “type” of the distribution function of \( V = \sum_{i=1}^{\infty} b^{-i} Y_i \) where \( \{Y_n, n \geq 1\} \) is a sequence of i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2} \) and \( b > 1 \). The distribution function \( F_V \) is known as an infinite Bernoulli convolution. There is an interesting literature of investigation on this problem going back to the groundbreaking work of Jessen and Wintner [12] and Erdös [8]. Jessen and Wintner [12] showed that \( F_V \) cannot be discrete and it cannot be a mixture; it must be either continuous singular or absolutely continuous. Erdös [8] showed that \( F_V \) is continuous singular if \( b > 2 \). A review of earlier developments concerning this problem is contained in Garsia [9].

Garsia [9] also pointed out several references of \( V \) and \( F_V \) arising in connection with psychological experimentation or in data transmission problems. We refer the reader to the more recent work of Diaconis and Freedman [5] for a concise discussion of this problem of identifying the “type” of \( F_V \); this work contains a brief discussion of the substantial recent discovery of Solomyak [18] (whose proof was subsequently simplified by Peres and Solomyak [13]) asserting that \( F_V \) is absolutely continuous for almost every [Lebesgue measure] value of \( b \in (1, 2] \). (Solomyak’s [18] result had previously been conjectured by Garsia [9].) The exceptional set is nonempty. For example, Erdös [8] showed for \( b = (1 + \sqrt{5})/2 \) (= the golden ratio \( \varphi \)) that \( F_V \) is continuous singular. The problem of characterizing the “type” of the distribution function \( F_V \) of \( V = \sum_{i=1}^{\infty} b^{-i} Y_i \) where \( \{Y_n, n \geq 1\} \) is a sequence of nondegenerate i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = 1 - \mathbb{P}(Y_1 = -1) \neq \frac{1}{2} \) was investigated by Peres and Solomyak [14].

The second example shows that, in general, the second inclusion in (3.1) of Theorem 3.1 is also proper; we exhibit a point \( c \in (l, L) \) such that

\[
\mathbb{P}(\{\omega \in \Omega : c \text{ is a limit point of } W_n(\omega)\}) = 0.
\]

**Example 4.2.** Let \( \{Y_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( \mathbb{P}(Y_1 = 0) = \mathbb{P}(Y_1 = 1) = \frac{1}{2} \). Then \( l = \operatorname{ess inf} Y_1 = 0 \) and \( L = \operatorname{ess sup} Y_1 = 1 \). Let \( b > 2 \). Note that for almost every \( \omega \in [Y_n, 0] \) where \( n \geq 2 \)

\[
W_n(\omega) = \frac{\sum_{i=1}^{n-1} b^i Y_i(\omega)}{b^{n+1}/(b-1)} \leq \frac{(b-1) \sum_{i=1}^{n-1} b^i}{b^{n+1}} = \frac{b^n - b}{b^{n+1}} < \frac{1}{b}.
\]

and that for almost every \( \omega \in [Y_n, 1] \) where \( n \geq 2 \)

\[
W_n(\omega) \geq \frac{(b-1)b^n}{b^{n+1}} = \frac{b-1}{b}.
\]

Thus, for all \( n \geq 2 \),

\[
\mathbb{P}\left( W_n < \frac{1}{b} \text{ or } W_n \geq \frac{b-1}{b} \right) = 1. \tag{4.1}
\]
Now, since $b > 2$,
\[ \frac{1}{b} < \frac{b - 1}{b} \]
and so letting
\[ c \in \left( \frac{1}{b}, \frac{b - 1}{b} \right) \subseteq (l, L) \quad \text{and} \quad \varepsilon \in \left( 0, \left( c - \frac{1}{b} \right) \wedge \left( \frac{b - 1}{b} - c \right) \right), \]
it follows from (4.1) that for all $n \geq 2$
\[ \mathbb{P} (W_n \in (c - \varepsilon, c + \varepsilon)) = 0 \]
and hence
\[ \mathbb{P} (W_n \in (c - \varepsilon, c + \varepsilon) \text{ i.o. (n)}) = 0. \]
It follows that
\[ \liminf_{n \to \infty} |W_n - c| \geq \varepsilon \quad \text{a.s.} \]
and so
\[ \mathbb{P} \left( \{ \omega \in \Omega : c \text{ is a limit point of } W_n(\omega) \} \right) = 0. \]

**Remark 4.2.** In Example 4.2, if $b = 3$, then $V_n = 2 \sum_{i=1}^{n} 3^{-i}Y_i \to V$ a.s. where $F_V$ is the distribution function of the Cantor singular distribution (see, e.g., Billingsley [2, p. 416]). Thus, $C = S(F_V)$ is the Cantor ternary set.

**References**


