The Convergence Rate for the Normal Approximation of Extreme Sums

Yongcheng Qi

University of Minnesota Duluth

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This talk is based on

- a joint work with Professor Shihong Cheng

Outline

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- Introduction
- Main Results
- Sketch of Proofs

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Extreme Sum

- Let $\{X, X_1, X_2, \cdots\}$ be a sequence of independent and identically distributed random variables with distribution F
- For each $n \geq 1$ let $X_{n,1} \leq \cdots \leq X_{n,n}$ denote the order statistics of $X_1, \cdots, X_n$
- Define $S_{n,k} = \sum_{j=1}^{k} X_{n,n-j-1}$ as an extreme sum, where $k = k_n$ satisfies

$$\lim_{n\to\infty} k_n = \infty \text{ and } \lim_{n\to\infty} \frac{k_n}{n} = 0 \qquad (1)$$

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Why extreme sums?

- Influence of extremes sums in the partial sums $\sum_{j=1}^{n} X_j$
  –Trimmed sums or extreme sums

- Estimation of the tail of a distribution in extreme-value statistics
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Domain of Attraction of An EV Distribution

- Assume that F belongs to the domain of attraction of an extreme value distribution, i.e. there exist constants $c_n > 0$ and $d_n \in R$ such that

$$\frac{X_{n,n} - d_n}{c_n} \xrightarrow{d} \exp\{-(1+\gamma x)^{-1/\gamma}\} =: G_\gamma(x) \qquad (2)$$

for some $\gamma \in R$.

- G_γ: extreme-value distribution
$\gamma > 0$: Fréchet
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Some Distributions

- Heavy-tailed distributions ($\gamma > 0$)

$$1 - F(x) = x^{-1/\gamma} L(x), x > 0$$

where $L(x)$ is a slowly varying function at infinity;

- Normal and exponential ($\gamma = 0$)
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Limiting Distribution of S_{n,k_n} under (2)

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Distributional Convergence Rate

- Mijnheer (1988) showed that

$$\sup_x |P(n^{-1/\alpha} S_{n,k_n} \leq x) - G(x;\alpha)| = O(\max(k_n^{1-1/\alpha}, n^{-1}))$$

if

$$1 - F(x) = x^{-\alpha} + O(x^{-r}) \quad x \to \infty$$

for some $r > \alpha$, $0 < \alpha < 1$, where $G(x;\alpha)$ is an asymmetric stable law.

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Von-Mises Conditions

- Denote $\omega_F = \sup\{x : F(x) < 1\}$.
- The following von-Mises conditions are sufficient for (2): F has the derivative function F' satisfying

$$\gamma > 0: \quad \omega_F = \infty, \lim_{x\to\infty} \frac{xF'(x)}{1 - F(x)} = \frac{1}{\gamma};$$

$$\gamma < 0: \quad \omega_F < \infty, \lim_{x\to\infty} \frac{(\omega_F - x)F'(x)}{1 - F(x)} = -\frac{1}{\gamma};$$

$$\gamma = 0: \quad \lim_{x\to\omega_F} \frac{F'(x)}{[1 - F(x)]^2} \int_x^{\omega_F} [1 - F(u)]du = 1.$$

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Von-Mises Conditions

- Let $V(t) = \inf\{x : [1 - F(x)]^{-1} \geq t\}$ for $t \geq 1$
- These von-Mises conditions are equivalent to the assumption that V has a derivative function $V' \in Rv(\gamma - 1)$, where $Rv(\gamma - 1)$ denotes the class of all regularly varying functions with index $\gamma - 1$, i.e.

$$\lim_{t\to\infty} \frac{V'(tx)}{V'(t)} = x^{\gamma-1}, \quad \forall x > 0. \tag{3}$$

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Notation

- Φ, ϕ – the standard normal distribution function and its density
- D^2X – the variance of a random variable X;
  Z – a random variable with distribution function $F_Z(x) = 1 - x^{-1}$, $x \geq 1$.
- Set

$$a_n = \frac{n - k_n + 1}{n + 1}; \quad b_n = \sqrt{\frac{a_n(1 - a_n)}{n + 1}};$$

$$l_n = \frac{1}{1 - a_n}; \quad \tau_n = \frac{\sqrt{a_n}(EV(l_n Z) - V(l_n))}{DV(l_n Z)}$$

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We are going to estimate the convergence rate

$$\Delta_n(x) = P\Big(\frac{S_{n,k_n} - k_n EV(I_n Z)}{\sqrt{k_n(1+\tau_n^2)D^2 V(I_n Z)}} \leq x\Big) - \Phi(x)$$

Theorem 1

Theorem

Under (1), if (3) holds for some $\gamma \in [1/3, 1/2)$, *then*

$$\sup_{x \in R} |\Delta_n(x)| = o(k_n^{-\epsilon}) \tag{4}$$

holds for any

$$\epsilon \in (0, (2\gamma)^{-1} - 1). \tag{5}$$

Theorem 2

Theorem

Under (1), if (3) holds for $\gamma < 1/3$, *then*

$$\Delta_n(x) = \frac{1}{\sqrt{k_n}} \frac{2(1-2\gamma)^{1/2}}{6\big(2(1-\gamma)\big)^{3/2}(1-3\gamma)} \Big(\Big(\xi_1(\gamma) + \xi_2(\gamma)x^2\Big)\phi(x) + \Phi_\gamma(x)\Big) + o\Big(\frac{1}{\sqrt{k_n}}\Big) \quad (6)$$

holds uniformly on $x \in R$, *where*

$$\xi_1(\gamma) = 7 - 8\gamma + 9\gamma^2, \quad \xi_2(\gamma) = 1 - 16\gamma - 15\gamma^2 + 26\gamma^3$$
$$\Phi_\gamma(x) = 2(1-2\gamma)(1-3\gamma)\int_{-\infty}^{x}(3u - u^3)\phi(u)du.$$

Sketch of the proofs

- Express the extreme sum as a sum of k_n conditionally iid RVs
- Estimate the moments of these RVs
  –by using the regularity of V and V'
- Find normal approximate error for the distribution of the sum of these conditionally iid RVs

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Some Notation

- Let $Z_1, Z_2, \cdots$ be independent random variables having the same distribution function as Z and let $Z_{n,1} \leq \cdots \leq Z_{n,n}$ be the order statistics of $Z_1, \cdots, Z_n$. Then

$$(X_{n,1}, \cdots, X_{n,n}) \stackrel{d}{=} (V(Z_{n,1}), \cdots, V(Z_{n,n})).$$

- For every $u \in R$ set

$$a_n(u) = \{(a_n + b_n u) \wedge 1\} \vee 0; \quad l_n(u) = (1 - a_n(u))^{-1};$$

$$S_{n,k_n}(u) = \sum_{j=1}^{k_n} V(l_n(u) Z_j); \;\; \phi_n(u) = \phi(u)(1 - \frac{u^3 - 3u}{3\sqrt{k_n}}).$$

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Fact 1:

Under (1), we have

$$\sup_{x\in R}\left|P(S_{n,k_n}\le x)-\int_{-\infty}^{\infty}P(S_{n,k_n}(u)\le x)\phi_n(u)du\right| = o\Big(\frac{1}{\sqrt{k_n}}\Big). \tag{7}$$

Fact 2:

If (3) holds for $\gamma \in R$, then

$$\lim_{t\to\infty} \frac{V(tx) - V(t)}{tV'(t)} = \frac{x^\gamma - 1}{\gamma}, \quad \forall x > 0 \tag{8}$$

with convention $\left.\frac{x^\gamma - 1}{\gamma}\right|_{\gamma=0} = \log x$ for $x > 0$. Furthermore, for any $\delta > 0$, there exists a $t_\delta > 0$ such that

$$(1-\delta)\frac{x^{\gamma-\delta} - 1}{\gamma - \delta} \le \frac{V(tx) - V(t)}{tV'(t)} \le (1+\delta)\frac{x^{\gamma+\delta} - 1}{\gamma + \delta} \tag{9}$$

holds for all $x \ge 1$ and $t \ge t_\delta$.

Fact 3:

Under (1) and (3), we have

$$\lim_{t\to\infty}\int_1^\infty \left[\frac{V(tv)-V(t)}{tV'(t)}\right]^j \frac{dv}{v^2} = \int_1^\infty \frac{(v^\gamma-1)^j}{\gamma^j v^2}dv, \quad \forall \gamma < j^{-1} \tag{10}$$

for all $j = 1, 2, \cdots$.

Some estimates

We conclude

- $$\lim_{n\to\infty} \frac{EV(l_n Z) - V(l_n)}{l_n V'(l_n)} = \int_1^\infty \frac{v^\gamma - 1}{\gamma} \frac{dv}{v^2} = \frac{1}{1-\gamma} \text{ if } \gamma < 1. \quad (11)$$

- $$\lim_{n\to\infty} \frac{DV(l_n Z)}{l_n V'(l_n)} = \frac{1}{(1-\gamma)(1-2\gamma)^{1/2}} =: \sigma(\gamma) \quad \text{if } \gamma < 1/2. \quad (12)$$

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Fact 4:

If $\gamma < 1/2$, then under (1) and (3), the following equations hold uniformly on $u \in [-k_n^{1/6}, k_n^{1/6}]$:

$$V(l_n(u)) - V(l_n) = \frac{l_n V'(l_n) u[1 + o(1)]}{\sqrt{k_n}}; \tag{13}$$

$$EV(l_n(u)Z) - EV(l_nZ) = \frac{u\tau_n DV(l_nZ)}{\sqrt{k_n}} + \frac{(1+\gamma) l_n V'(l_n) u^2 [1 + o(1)]}{2k_n(1-\gamma)}; \tag{14}$$

$$\frac{DV(l_nZ)}{DV(l_n(u)Z)} - 1 = -\frac{u[\gamma + o(1)]}{\sqrt{k_n}}. \tag{15}$$

Fact 5:

Suppose that (1) and (3) hold. If $\gamma < 1/2$, then there exist two functions $\alpha_{n,1}(u)$ and $\alpha_{n,2}(u)$ such that

$$\begin{aligned} \eta_n(x,u) &:= \frac{DV(I_nZ)x - \sqrt{k_n}[EV(I_n(u)Z) - EV(I_nZ)]}{DV(I_n(u)Z)} \quad (16) \\ &= x - \tau_n u - \alpha_{n,1}(u)xu - \alpha_{n,2}(u)u^2 \end{aligned}$$

holds for all $x \in R$ and $u \in [-k_n^{1/6}, k_n^{1/6}]$, where

$$\lim_{n\to\infty} \sqrt{k_n}\alpha_{n,1}(u) = \gamma \quad \text{and} \quad \lim_{n\to\infty} \sqrt{k_n}\alpha_{n,2}(u) = [2\sigma(\gamma)]^{-1}$$

uniformly on $[-k_n^{1/6}, k_n^{1/6}]$.

For $n = 1, 2, \cdots$ and $x, u \in R$, denote

$$\hat{Z}_{n,j}(u) = \frac{V(l_n(u)Z_j) - EV(l_n(u)Z)}{\sqrt{D^2 V(l_n(u)Z)}}, \quad j = 1, \cdots, k_n;$$

$$\hat{S}_{n,k_n}(u) = \frac{1}{\sqrt{k_n}} \sum_{j=1}^{k_n} \hat{Z}_{n,j}(u).$$

Proof of Theorem 1.

Denote

$$\Delta_{n,1} = \sup_{x\in R}\left|\int_{|u|>k_n^{1/6}}[P(\hat{S}_{n,k_n}(u)\le \eta_n(x,u))-\Phi(x-\tau_n u)]\phi(u)du\right|;$$

$$\Delta_{n,2} = \sup_{x\in R}\left|\int_{-k_n^{1/6}}^{k_n^{1/6}}[P(\hat{S}_{n,k_n}(u)\le \eta_n(x,u))-\Phi(\eta_n(x,u))]\phi(u)du\right|;$$

$$\Delta_{n,3} = \sup_{x\in R}\left|\int_{-k_n^{1/6}}^{k_n^{1/6}}[\Phi(\eta_n(x,u))-\Phi(x-\tau_n u)]\phi(u)du\right|.$$

Since

$$\Phi\Big(\frac{x}{\sqrt{1+\tau_n^2}}\Big) = \int_{-\infty}^{\infty} \Phi(x \pm \tau_n u)\phi(u)du, \tag{17}$$

we have from (7) and (16) that

$$\begin{aligned}
\sup_{x\in R} |\Delta_n(x)| &= \sup_{x\in R} \Big|\Delta_n\Big(\frac{x}{\sqrt{1+\tau_n^2}}\Big)\Big| \\
&\leq \Big|\sup_{x\in R} \int_{-\infty}^{\infty} [P(\hat{S}_{n,k_n}(u) \leq \eta_n(x,u)) - \Phi(x - \tau_n u)]\phi(u)du| \\
&\quad + \frac{C}{\sqrt{k_n}} \\
&\leq \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3} + \frac{C}{\sqrt{k_n}}.
\end{aligned}$$

- $\Delta_{n,1} \leq P(|N| > k_n^{1/6}) = O(k_n^{-1/2})$;
- From the Berry-Esseen theorem we get for any $\epsilon \in (0, 1/2]$

$$\begin{aligned}\Delta_{n,2} &\leq \sup_{|u| \leq k_n^{1/6}} \sup_{x \in R} |P(\hat{S}_{n,k_n}(u) \leq \eta_n(x,u)) - \Phi(\eta_n(x,u))| \\ &\leq Ck_n^{-\epsilon}\end{aligned}$$

- $\Delta_{n,3} = O(k_n^{-1/2})$.

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Proof of Theorem 2

For $n = 1, 2, \cdots$ and $t, x, u \in R$, let

$$\begin{aligned}
\rho(\gamma) &= \frac{2(1+\gamma)\sqrt{1-2\gamma}}{1-3\gamma};\\
G(x) &= \Phi(x) - \frac{\rho(\gamma)(x^2-1)\phi(x)}{3\sqrt{k_n}};\\
F_n(x,u) &= P(\hat{S}_{n,k_n}(u) \le x);\\
f_n(t,u) &= E\exp\{it\hat{Z}_{n,1}(u)\}\\
&\qquad \text{(the characteristic function of } \hat{Z}_{n,1}(u));\\
\psi_n(t,u) &= \log f_n\Big(\frac{t}{\sqrt{k_n}}, u\Big) + \frac{t^2}{2k_n}.
\end{aligned}$$

Fact 6:

Suppose (1) and (3) hold. Then

$$\lim_{n\to\infty} \sup_{|u|\le k_n^{1/6}} |E\hat{Z}_{n,1}^3(u) - \rho(\gamma)| = 0, \tag{18}$$

and for any positive p with $\gamma p < 1 - 3\gamma$

$$\limsup_{n\to\infty} \sup_{|u|\le k_n^{1/6}} E|\hat{Z}_{n,1}(u)|^{3+p} < \infty. \tag{19}$$

Fact 7:

Under (1) and (3), we have

$$\sup_{|u|\leq k_n^{1/6}} \sup_{x\in R} |F_n(x,u) - G(x)| = o\left(\frac{1}{\sqrt{k_n}}\right). \tag{20}$$

Proof of Theorem 2.

We get

$$\begin{aligned}
& \Delta_n\Big(\frac{x}{\sqrt{1+\tau_n^2}}\Big) \\
= \quad & \int_{-k_n^{1/6}}^{k_n^{1/6}} [\Phi(\eta_n(x,u)) - \Phi(x-\tau_n u)])\phi_n(u)du \\
& -\frac{\rho(\gamma)}{3\sqrt{k_n}} \int_{-k_n^{1/6}}^{k_n^{1/6}} [\eta_n^2(x,u)) - 1]\phi(\eta_n(x,u))\phi_n(u)du \\
& + \int_{-k_n^{1/6}}^{k_n^{1/6}} \Phi(x-\tau_n u))[\phi_n(u) - \phi(u)]du + o\Big(\frac{1}{\sqrt{k_n}}\Big) \\
=: \quad & K_{n,1}(x) + K_{n,2}(x) + K_{n,3}(x) + o\Big(\frac{1}{\sqrt{k_n}}\Big)
\end{aligned}$$

holds uniformly on $x \in R$.

Thank you very much!

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