CHAPTER 2

PROBABILITY

2.1 Sample Space

A *probability model* consists of the sample space and the way to assign probabilities.

Sample space & sample point

The *sample space* S, is the set of all possible outcomes of a statistical experiment.

Each outcome in a sample space is called a *sample point*. It is also called an *element* or a *member* of the sample space.

For example, there are only two outcomes for tossing a coin, and the sample space is

$$S = \{\text{heads, tails}\}, \quad \text{or}, \quad S = \{\text{H, T}\}.$$

If we toss a coin three times, then the sample space is

$$S = \{\text{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT}\}.$$

EXAMPLE 2.1. Consider rolling a fair die twice and observing the dots facing up on each roll. What is the sample space?

There are 36 possible outcomes in the sample space S, where

$$S = \left\{ \begin{array}{cccccc} (1,\ 1) & (1,\ 2) & (1,\ 3) & (1,\ 4) & (1,\ 5) & (1,\ 6) \\ (2,\ 1) & (2,\ 2) & (2,\ 3) & (2,\ 4) & (2,\ 5) & (2,\ 6) \\ (3,\ 1) & (3,\ 2) & (3,\ 3) & (3,\ 4) & (3,\ 5) & (3,\ 6) \\ (4,\ 1) & (4,\ 2) & (4,\ 3) & (4,\ 4) & (4,\ 5) & (4,\ 6) \\ (5,\ 1) & (5,\ 2) & (5,\ 3) & (5,\ 4) & (5,\ 5) & (5,\ 6) \\ (6,\ 1) & (6,\ 2) & (6,\ 3) & (6,\ 4) & (6,\ 5) & (6,\ 6) \end{array} \right\}$$

NOTE. You may use a **tree diagram** to systematically list the sample points of the sample space.

EXAMPLE 2.2. A fair six-sided die has 3 faces that are painted blue (**B**), 2 faces that are red (**R**) and 1 face that is green (**G**). We toss the die twice. List the complete sample space of all possible outcomes.

(a) if we are interested in the **color** facing upward on each of the two tosses.

(b) if the outcome of interest is the **number** of red we observe on the two tosses.

NOTE. A **statement** or **rule method** will best describe a sample space with a large or infinity number of sample points. For example, if S is the set of all points (x, y) on the boundary or the interior of a unit circle, we write a rule/statement $S = \{(x, y) | x^2 + y^2 \leq 1\}$.

EXAMPLE 2.3. List the elements of each of the following sample spaces:

(a) $S = \{x | x^2 - 3x + 2 = 0\}$

(b) $S = \{x | e^x < 0\}$

NOTE. The **null set**, or empty set, denoted by ϕ, contains no members/elements at all.

2.2 Events

Event

An *event* is a subset of a sample space.

Refer to Example 2.1. "the sum of the dots is 6" is an event. It is expressible of a set of elements

$$E = \{(1,\ 5) \quad (2,\ 4) \quad (3,\ 3) \quad (4,\ 2) \quad (5,\ 1)\}$$

Complement

The event that A does not occur, denoted as A', is called the complement of event A.

EXAMPLE 2.4. Refer to Example 2.1. What are the complement events of

(a) event A "the sum of the dots is greater than 3"

(b) event B "the two dots are different"

Intersection

The *intersection* of two events A and B, denoted by $A \cap B$, is the event containing all elements that are common to A and B.

EXAMPLE 2.5. Refer to the preceding example. Give $A \cap B'$, $A' \cap B$ and $A' \cap B'$.

Mutually Exclusive

Events that have no outcomes in common are said to be *disjoint* or *mutually exclusive*.

Clearly, A and B are mutually exclusive or disjoint if and only if $A \cap B$ is a null set.

EXAMPLE 2.6. Refer to Example 2.1. Consider the events

$$C = \{\text{first die faces up with a } 6\}$$
$$D = \{\text{sum is at most } 5\}.$$

Verify that C and D are mutually exclusive.

Union

The *union* of the two events A and B, denoted by $A \cup B$, is the event containing all the elements that belong to A or B or both.

EXAMPLE 2.7. Refer to Example 2.1. Consider the events

$$B = \{\text{the two dots are different}\}$$
$$F = \{\text{sum is at least } 11\}.$$

What is $B' \cup F$?

NOTE. The **Venn diagram** is a useful tool to graphically illustrate the complement, intersection and union of the events.

EXAMPLE 2.8. In a school of 320 students, 85 students are in the band, 200 students are on sports teams, and 60 students participate in both activities. How many students are involved in either band or sports? How many are involved neither band nor sports?

EXAMPLE 2.9. A guidance counselor is planning schedules for 30 students. Sixteen students say they want to take French, 16 want to take Spanish, and 11 want to take Latin. Five say they want to take both French and Latin, and of these, 3 wanted to take Spanish as well. Five want only Latin, and 8 want only Spanish. How many students want French only?

Useful relationships

$$A \cap \phi = \phi \qquad S' = \phi$$
$$A \cup \phi = A \qquad \phi' = S$$
$$A \cap A' = \phi \qquad (A \cap B)' = A' \cup B'$$
$$A \cup A' = S \qquad (A \cup B)' = A' \cap B'$$
$$(A')' = A$$

2.3 Counting Sample Points

Multiplication rule

If an operation can be performed in n_1 ways, and if of each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

NOTE. This rule can be extended for k operations. It is called the *generalized multiplication rule*.

EXAMPLE 2.10. (a) A business man has 4 dress shirts and 7 ties. How many different shirt/tie outfits can he create?

(b) How many sample points are in the sample space when a coin is flipped 4 times?

EXAMPLE 2.11. How many even three-digit numbers can be formed from the digits 0, 3, 4, 7, 8, and 9 if each digit can be used only once?

Permutation

A *permutation* is an arrangement of all or part of a set of objects. *The arrangements are different/distinct.*

Denote by *factorial symbol* ! the product of decreasing positive whole numbers.

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

By convention, $0! = 1$.

Factorial rule

The number of permutations of n objects is $n!$.

That is, a collection of n different items can be arranged in order $n! = n(n-1)\cdots 2 \cdot 1$ different ways.

Permutation rule (when all items are all different)

The number of permutations of n distinct objects taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

That is to say, there are $n(n-1)\cdots(n-r+1)$ ways to select r items from n available items **without** replacement.

EXAMPLE 2.12. If there are 27 shows and 4 time slots on Thursday. How many different sequences of 4 shows are possible from the 27 available?

Circular permutation

The number of permutations of n objects arranged in a circle is $(n-1)!$.

Permutation rule (when some items are identical to others)

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, ..., n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

EXAMPLE 2.13. How many different ways can we arrange: $S\,T\,A\,T\,I\,S\,T\,I\,C$?

If we consider the **combination**, i.e., partitioning a set of n objects into r cells/subsets and *the order of the elements within a cell is of* <u>*no*</u> *important*, then the following rule applies

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cells, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

Combinations rule

The number of combinations of n distinct objects taken r at a time

$${}_nC_r = \binom{n}{r, n-r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{{}_nP_r}{r!}$$

NOTE. (1) We must have a total of n different items available. (2) We must select r of the n items (without replacement). (3) We must consider rearrangements of the same items to be the same. (The combination ABC is the same as CBA.)

EXAMPLE 2.14. Determine the total number of five-card hands that can be drawn from a deck of 52 ordinary playing cards.

Permutations versus Combinations

When different orderings of the same items are to be *counted separately*, we have a permutation problem; but when different orderings are *not to be counted separately*, we have a combination problem.

EXAMPLE 2.15. In horse racing, a trifecta is a type of bet. To win a trifecta bet, you need to specify the horses that finish in the top three spots in the exact order in which they finish. If eight horses enter the race, how many different ways can they finish in the top three spots?

EXAMPLE 2.16. There are 18 faculty members in the department of Mathematics and Statistics. Four people are to be in the executive committee. Determine how many different ways this committee can be created.

EXAMPLE 2.17. In a local election, there are seven people running for three positions. The person that has the most votes will be elected to the highest paying position. The person with the second most votes will be elected to the second highest paying position, and likewise for the third place winner. How many different outcomes can this election have?

EXAMPLE 2.18. The director of a research laboratory needs to fill a number of research positions; two in biology and three in physics. There are seven applicants for the biology positions and 9 for the physicist positions. How many ways are there for the director to select these people?

2.4 Probability of an Event

Let $\mathsf{P}(A)$ denote the probability that event A occurs.

Any probability is a number between 0 and 1.

For any event A,

$$0 \le \mathsf{P}(A) \le 1$$

It is trivial to see that

$$\mathsf{P}(\phi) = 0 \qquad \text{and} \qquad \mathsf{P}(S) = 1$$

Ways to assign the probabilities

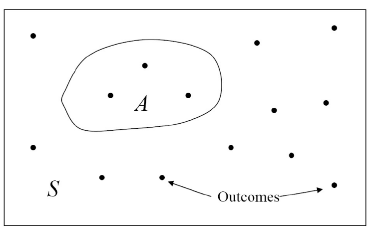
If the sample space S consists of a finite (or countable infinite) number of outcomes, assign a probability to each outcome.

- The sum of all probabilities equals to 1.

- If there is k (finite) outcomes in the sample space S, **all equally likely**, then each individual outcome has probability $1/k$. The probability of event A is

$$\mathsf{P}(A) = \frac{\text{count of outcomes in } A}{\text{count of outcomes in } S}$$

Sample space S and event A



EXAMPLE 2.19. We roll two balanced dice and observe on each die the number of dots facing up. Determine the probability of the event that

(a) the sum of the dice is 6.

(b) doubles are rolled – that is, both dice come up the same number.

EXAMPLE 2.20. An American roulette wheel contains 38 numbers, of which 18 are red, 18 are black, and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. Determine the probabilities that the number on which the ball lands is red, black and green, respectively?

EXAMPLE 2.21. (a) Suppose that we select a card at random from a deck of 52 playing cards. Find the probability that the card selected is a red card. What about a spade card? A face card?

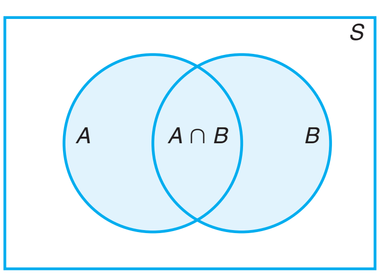
(b) In a poker hand of 5 cards, find the probability that the cards selected are 3 aces and 2 kings.

2.5 Additive Rules

Formal addition rule (General addition rule) for unions of two events

For any events A and B,

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$$



EXAMPLE 2.22. Suppose that we select a card at random from a deck of 52 playing cards. Find the probability that the card selected is either a spade or face card.

EXAMPLE 2.23. Let A and B be two events such that $\mathrm{P}(A \cup B) = 0.6$, $\mathrm{P}(A) = 0.5$, and $\mathrm{P}(B) = 0.3$. Determine $\mathrm{P}(A \cap B)$.

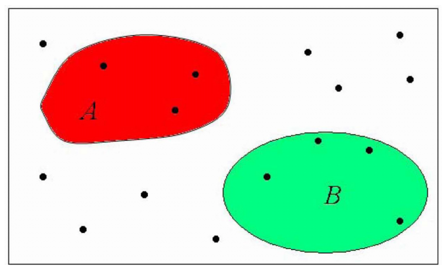
NOTE. For three events A, B and C,

$$\begin{aligned}\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) \\ - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) \\ + \mathrm{P}(A \cap B \cap C).\end{aligned}$$

Intuitive addition rule (Addition rule for mutually exclusive events)

If A and B are mutually exclusive,

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$$



EXAMPLE 2.24. Suppose that we select a card at random from a deck of 52 playing cards. Find the probability that the card selected is either an ace or a jack.

A useful formula for $\mathsf{P}(A \cap B')$

$$\mathsf{P}\left(A \cap B'\right) = \mathsf{P}(A) - \mathsf{P}(A \cap B)$$

EXAMPLE 2.25. Suppose that that $\mathrm{P}(A \cap B) = 0.1$, $\mathrm{P}(A) = 0.5$, and $\mathrm{P}(B) = 0.3$. Determine

(a) $\mathrm{P}\left(A \cap B'\right)$

(b) $\mathrm{P}\left(A' \cap B\right)$

NOTE. We define a **partition** $\{A_1, A_2, \ldots, A_n\}$ of a sample space S as a collection of events satisfying that

i) $A_1, A_2, \ldots, A_n$ are mutually exclusive; and

ii) $A_1 \cup A_2 \cup \cdots \cup A_n = S$.

It is easy to see that, for a partition $\{A_1, A_2, \ldots, A_n\}$ of S,

$$\begin{aligned}\mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_n) &= \mathrm{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= \mathrm{P}(S) \\ &= 1\end{aligned}$$

EXAMPLE 2.26. Let $S = \{1, 2, 3, 4, 5\}$. Determine if each of the following form a partition of S.

(a) $A = \{1, 3, 5\}$ and $B = \{2, 3, 4\}$

(b) $A = \{1, 3, 5\}$ and $C = \{2, 4\}$

Complementary rule

The probability that *event A does not occur* is 1 minus the probability that it occurs.

$$\mathsf{P}\left(A'\right) = 1 - \mathsf{P}(A)$$

NOTE. Clearly, $\mathsf{P}\left(A \cap A'\right) = 0$ and $\mathsf{P}\left(A \cup A'\right) = \mathsf{P}(S) = 1$. Thus, A and A' form a partition of S.

EXAMPLE 2.27. Refer to Example 2.20. Use the complementary rule to determine the probability that the number on which the ball lands is not black.

EXAMPLE 2.28. We choose a new car at random and record its color. Here are the probabilities of the most popular colors for cars made in North America.

Color	Silver	White	Black	Blue	Red
Probability	0.21	0.16	0.11	0.12	0.10

What is the probability that a randomly chosen car has any color *other than* the listed?

2.6 Conditional probability

Denote by $\mathsf{P}(B|A)$ the probability of event B occurring after it is assumed that event A has already occurred. It reads "the probability of B given A".

EXAMPLE 2.29. A jar consists of 5 sweets. 3 are green and 2 are blue. Lucky Smart wants to randomly pick two sweets without replacement. What is the probability that the second picked sweet is blue if it is known that her first pick is blue?

Conditional probability

A conditional probability $\mathsf{P}(B|A)$ can be found by dividing the probability of events A and B both occurring by the probability of event A

$$\mathsf{P}(B|A) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(A)}, \qquad \text{provided} \quad \mathsf{P}(A) > 0.$$

EXAMPLE 2.30. The probability that an automobile being filled with gasoline needs an oil change is 0.25; the probability that it also needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

(a) If the oil has to be changed, what is the probability that a new oil filter is needed?

(b) If a new oil filter is needed, what is the probability that the oil has to be changed?

EXAMPLE 2.31. Evidence from a variety of sources suggests that diets high in salt are associated with risks to human health. To investigate the relationship between salt intake and stroke, information from 14 studies were combined in a meta-analysis. Subjects were classified based on the amount of salt in their normal diet. They were followed for several years and then classified according to whether or not they had developed cardiovascular disease (CVD). Here are the data from one of the studies:

	Low salt	High salt
CVD	88	112
No CVD	1081	1134
Total	1169	1246

(a) If a subject has not developed CVD, what is the probability that he/she is from low salt group?

(b) If a subject is chosen at random from high salt group, what is the probability that he/she develops CVD?

(c) What is the probability that a randomly chosen subject develops CVD?

(d) What is the probability that a randomly chosen subject has developed CVD or he/she is from the low salt group?

Independent event

Two events are said to be *independent* if knowing one occurs does not change the probability of the other occurring. Otherwise, they are *dependent*.

That is,

A and B are independent

if and only if

$$\mathsf{P}(B|A) = \mathsf{P}(B)$$

EXAMPLE 2.32. Refer to Example 2.31. Based on parts (b) and (c), what can you say about the independence of "developing CVD" and "being in the high salt group"?

Formal (general) multiplication/product rule

For two events A and B,

$$\mathsf{P}(A \cap B) = \mathsf{P}(A)\,\mathsf{P}(B|A), \quad \text{provided } \mathsf{P}(A) > 0.$$

It is also true that $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B)$ by symmetry.

EXAMPLE 2.33. The probability that an automobile being filled with gasoline needs an oil change is 0.25; the probability that it also needs a new oil filter is 0.40; and the probability that the automobile need a new filter, given that it also needs an oil change is 0.60. Find the probability that

(a) both the oil and the filter need changing.

(b) the oil needs changing, if the filter needs changing.

(c) at least one needs changing.

Intuitive multiplication/product rule (for independent events)

Two events A and B are independent

if and only if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

EXAMPLE 2.34. A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight? What about a club and then an ace?

NOTE. The multiplication/product rule can be generalized for two or more events.

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \mathrm{P}(A_1)\mathrm{P}(A_2|A_1)\mathrm{P}(A_3|A_1 \cap A_2) \cdots \mathrm{P}(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

Clearly, if the events $A_1, A_2, \ldots, A_k$ are independent, then

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)\cdots\mathrm{P}(A_k).$$

EXAMPLE 2.35. Three cards are randomly selected form a deck of 52 ordinary playing cards. What is the probability of having three hearts? Consider both with- and without-replacement cases.

2.7 Bayes' Rule

***Theorem* (Total Probability).** If the events $B_1, B_2, \ldots, B_k$ form a partition of the sample space S such that $\mathrm{P}(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$\mathrm{P}(A) = \sum_{i=1}^{k} \mathrm{P}(B_i \cap A) = \sum_{i=1}^{k} \mathrm{P}(B_i)\mathrm{P}(A|B_i)$$

EXAMPLE 2.36. In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. What is the probability that a randomly selected finished product is defective?

Bayes' rule (general)

If the events B_1, B_2, ..., B_k form a partition of the sample space S such that $\mathrm{P}(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S such that $\mathrm{P}(A) \neq 0$, for $r = 1, 2, \ldots, k$

$$\mathrm{P}(B_r|A) = \frac{\mathrm{P}(B_r \cap A)}{\sum_{i=1}^{k} \mathrm{P}(B_i \cap A)} = \frac{\mathrm{P}(B_r)\mathrm{P}(A|B_r)}{\sum_{i=1}^{k} \mathrm{P}(B_i)\mathrm{P}(A|B_i)}.$$

NOTE (Bayes' Rule for two events). The probability of event E given that event A has subsequently occurred, is given by

$$\mathrm{P}(E|A) = \frac{\mathrm{P}(E)\ \mathrm{P}(A|E)}{\mathrm{P}(E)\ \mathrm{P}(A|E) + \mathrm{P}(E')\ \mathrm{P}(A|E')}$$

provided that $\mathrm{P}(A) > 0$ and $0 < \mathrm{P}(E) < 1$.

EXAMPLE 2.37. A class consists of 60% men and 40% women. Of the men, 25% are blond, while 45% of the women are blond. If a student is chosen at random and is found to be blond, what is the probability that student is a man?

EXAMPLE 2.38. A manufacturer claims that its drug test will detect steroid use (that is, show positive for an athlete who uses steroids) 95% of the time. Further, 15% of all steroid-free individuals also test positive. 10% of the rugby team members use steroids. Your friend on the rugby team has just tested positive. What is the probability that he uses steroids?

EXAMPLE 2.39. An aircraft emergency locator transmitter (ELT) is a device designed to transmit a signal in the case of a crash. The Altigauge Manufacturing Company makes 80% of the ELTs, the Bryant Company makes 15% of them, and the Chartair Company makes the other 5%. The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and the Chartair ELTs have a 9% rate of defects (which helps to explain why Chartair has the lowest market share).

(a) If an ELT is randomly selected from the general population of all ELTs, find the probability that it was made by the Altigauge Manufacturing Company.

(b) If a randomly selected ELT is then tested and is found to be defective, find the probability that it was made by the Altigauge Manufacturing Company.