

UNIVERSITY OF MINNESOTA

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AJIT MARATHE

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examining committee have been made.

Dr. Douglas Dunham

Name of Faculty Adviser

Signature of Faculty Adviser

August 6, 2007

Date

GRADUATE SCHOOL

**Incorporating Points at Infinity in a Hyperbolic
Drawing Program**

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AJIT MARATHE

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Dedicated to
my mom
Mrs. Suniti Marathe,
and my dad
Mr. Ashok Marathe

Abstract

The Poincaré disk model is one model for hyperbolic non-Euclidean geometry. It resides inside a circle called the "Boundary Circle". The inside of this boundary circle is the graphing area of the screen and it models the entire, infinite, two-dimensional hyperbolic plane. This geometry can be studied in a manner like Euclidean geometry, with axioms about points and lines, but as for Euclidean geometry, it is helpful to have physical models and models on which to draw.

In the past, to use the Poincaré model, we started with a Euclidean circle and points inside the circle. In this thesis we check for points on the circle i.e. points at infinity. They are not true points of the model. They represent directions at infinity. We use them to make some constructions in the model.

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Chapter 1

Introduction

Euclidean geometry is a mathematical system attributed to the Greek mathematician Euclid of Alexandria. It is an axiomatic system in which all theorems are derived from a finite number of axioms. Hyperbolic geometry is similar to Euclidean geometry in many respects. It has concepts of distances and angle and there are many theorems common to both. But there are also striking differences.

Euclidean geometry is usually the most convenient to describe the physical world around us. On the other hand hyperbolic geometry describes relativistic space-time more conveniently.

The primary goal of this thesis is to incorporate points at infinity into a hyperbolic drawing program to draw geometric objects.

Subsequent chapters will deal with basic principles of hyperbolic

geometry, its models, incorporating points at infinity, results, and directions of future research.

Chapter 2

Discovery of Hyperbolic Geometry

2.1 Euclidean Geometry

Euclidean geometry is a mathematical system attributed to the Greek mathematician Euclid of Alexandria. Euclid's text "*Elements*" was the first systematic discussion of geometry. It has been one of the most influential books in history, as much for its method as for its mathematical content. The method consists of assuming a small set of intuitively appealing axioms, and then proving many other propositions (theorems) from those axioms. Although many of Euclid's results had been stated by earlier Greek mathematicians, Euclid was the first to show how these propositions could be fitted together into a comprehensive deductive and logical system.

For over two thousand years, the adjective "Euclidean" was unnecessary because no other sort of geometry had been conceived. Euclid's axioms seemed so intuitively obvious that any theorem proved from them was

deemed true in an absolute sense. Today, however, many other self-consistent geometries are known, the first ones having been discovered in the early 19th century. It also is no longer taken for granted that Euclidean geometry described physical space. An implication of Einstein's theory of general relativity is that Euclidean geometry is only a good approximation to the properties of physical space if the gravitational field is not too strong.

Euclidean geometry is an axiomatic system, in which all theorems are derived from a finite number of axioms. Euclid gives five postulates (axioms)

- 1) Any two points can be joined by a straight line.
- 2) Any straight line segment can be extended indefinitely in a straight line.
- 3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- 4) All right angles are congruent.

5) Parallel Postulate: If two lines intersect a third line in such a way that the sum of the inner angles on one side is less than the two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

2.2 Non Euclidean Geometry

Euclid's fifth postulate had always been thought of mathematicians as special. Mathematicians never doubted its truth, but always thought of it as a theorem, which could be proved from the other four.

By the latter half of the 18th century, this problem of proving the fifth postulate had become really famous and many mathematicians had attempted to prove it. It was only a matter of time before the difficulty of the problem would cause some to conclude that this problem was unsolvable. It did not in any way mean that Postulate 5 was not provable. But this unproven idea made the discovery of non-Euclidean geometry inevitable. The logical transition behind it was that since neutral geometry, the

geometry was Euclid's first four axioms, itself did not imply Postulate 5, there must be a new geometry different from Euclid's based on the first four axioms and the negation of Postulate 5.

It has been observed many times in the history of science and mathematics, that when many people are working on the same problem, and when the communication between them is infrequent, it leads to multiple independent discoveries. In this sense, it seems that non-Euclidean geometry was discovered about four times in a span of twenty years. Apparently, Carl Friedrich Gauss was the first to discover non-Euclidean geometry. But he did not publish any papers related to his discoveries. He worked on this new geometry for many years and discovered many theorems.

In the meanwhile, he received a letter from Ferdinand Schweikart which indicated that Schweikart had himself discovered non-Euclidean geometry and had basically reached similar conclusions and results as Gauss.

Neither of them published any papers though, and hence, when Janós Bolyai published his work on non-Euclidean geometry in the Appendix of his father's book, it established the field of non-Euclidean geometry.

Bolyai was not the first person to have a paper published on non-Euclidean geometry. A Russian mathematics professor, Nicolai Lobachevsky had already published a paper on the topic. But since the paper was published in Russian, it was not well known in the European mathematical circles. Its translation into French and German later on reaffirmed Lobachevsky's work and discoveries in the field of non-Euclidean geometry.

Non-Euclidean geometry is technically any geometry which is not Euclidean. One of the most useful non-Euclidean geometries is the hyperbolic geometry, the geometry of hyperbolic space. This is the geometry discovered by Bolyai, Gauss, Lobachevsky and Schweikart [4]. Spherical or Elliptical geometry, the geometry of sphere, is another non-Euclidean geometry.

2.3 Hyperbolic Geometry

Hyperbolic geometry is defined as

The geometry you obtain by assuming all the axioms for neutral geometry (geometry without a parallel postulate) and replacing Hilbert's parallel postulate by its negation, which we call the "hyperbolic axiom". [2]

Basically, in hyperbolic geometry, all of Euclid's postulates hold, other than Postulate 5. Instead of the fifth postulate, an axiom called the *Hyperbolic axiom* is used.

According to the Hyperbolic axiom:

If there exists a line l and a point P not on the line l , there are at least two distinct lines parallel to l passing through P .

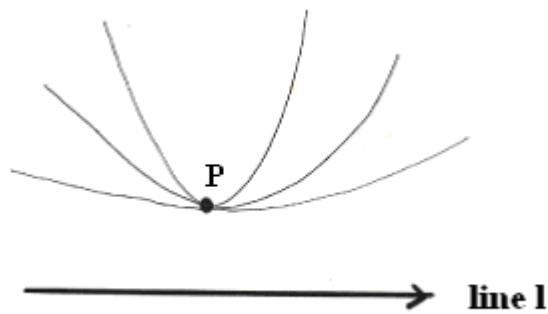


Figure 2.1 : Hyperbolic Axiom

This postulate in turn, is used to prove and obtain many other useful results including:

- There exists a triangle whose angle sum is less than 180° .
- All quadrilaterals have angle sum less than 360° .
- If two triangles are similar, they are congruent, that is, if the angles of the triangle are equal, so are the sides.

To prove that any geometry is consistent, we need a model of it. A model is an interpretation of the primitive terms under which the axioms become true statements. Here interpretation does not allude to the “understanding of the meaning”, but in a more basic sense of “giving a meaning”.

Unlike Euclidean geometry models that are infinite, some models of hyperbolic space can represent hyperbolic objects in a finite portion of Euclidean 2-space. The Beltrami-Klein model and the Poincaré circle model are examples of such finite models, while the Poincaré Upper half plane model, the Weierstrass model, and the Minkowski model are infinite structures embedded in Euclidean space. The finite models have boundaries,

which play an important role in the definition of parallel lines in hyperbolic space.

In the following sections, we discuss three of these models which are relevant to the present work.

Chapter 3

Models of Hyperbolic geometry

3.1 The Weierstrass Model

Weierstrass demonstrated the consistency of hyperbolic geometry by constructing a model for it on a certain surface in Euclidean 3-space [1].

We now provide a brief development of this model following [1]. We will look at the interpretation of the primitive terms like “point”, “line”, “lies on” and “between” given by it. Before that, we need some preliminaries.

A “point” in real 3-dimensional space is represented by a triplet of real numbers such as $X = (x, y, z)$. Such a triplet can also represent a vector in 3-space. This vector can be visualized as an arrow from the origin $O = (0, 0, 0)$ to X . This lets us use the terms “vector” and “point” interchangeably.

Consider two vectors $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$. Their *Euclidean inner product or dot product* is defined as:

$$X_1 \cdot X_2 = x_1x_2 + y_1y_2 + z_1z_2$$

To construct the model we also need another inner product, called the *hyperbolic inner product*. For the above two vectors it can be defined as

$$\langle X_1, X_2 \rangle = x_1x_2 + y_1y_2 - z_1z_2 \quad (3.1)$$

Two vectors X and Y are said to be *e-orthogonal* if $X \cdot Y = 0$ and *h-orthogonal* if $\langle X, Y \rangle = 0$.

Now the equation of the plane in R^3 is of the form $\langle A, X \rangle = b$, where A is an *h-normal* of the plane. This means that A is h-orthogonal to any point that lies on this plane.

We are now ready to develop the model. As noted earlier we provide interpretations of “point”, “line”, “lies on” and “between”. We will also use the symbol R^3 to mean the Euclidean 3-space.

A “point” is defined to be a point (or vector) $X = (x, y, z)$ of R^3 such that

$\langle X, X \rangle = -k^2$ and $z > 0$. Such points comprise the “upper sheet” of a two sheeted hyperboloid in R^3 . We call this sheet H^2 . The constant k is a measure of the curvature of the hyperbolic plane.

A “line” is defined to be the intersection of H^2 with a plane through the origin of R^3 . We know from Euclidean geometry that such intersection is one branch of a hyperbola (Fig. 3.1).

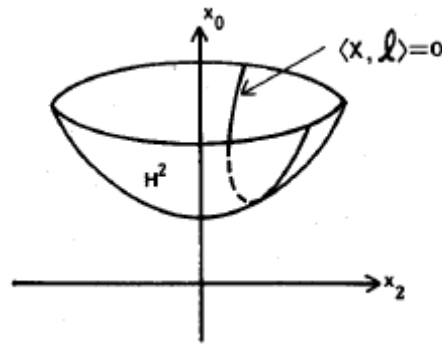


Figure 3.1 : A “line” in the Weierstrass Model [1]

Now, a plane through the origin is given by an equation of the form $\langle X, l \rangle = 0$, where l is an h-normal of the plane. Since l is an h-normal, any scalar multiple of l will also be an h-normal to the plane. We shall choose l in such a way that $\langle l, l \rangle = k^2$. This means that l is a point of the single-sheeted hyperboloid with the equation $\langle X, X \rangle = k^2$. (Fig. 3.2).

Summarizing, a “line” is a section of H^2 by a plane through the origin of R^3 . There is a vector l , such that $\langle l, l \rangle = k^2$ and the line consists of all X of H^2 such that $\langle X, l \rangle = 0$.

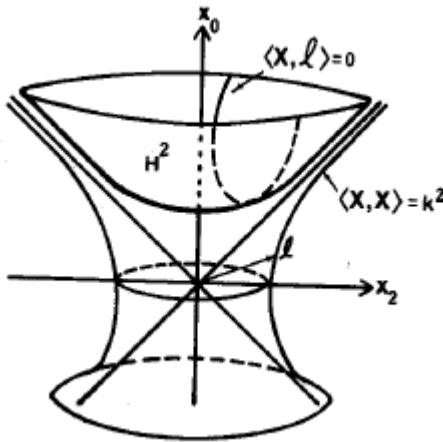


Figure 3.2 : The vector l on the hyperboloid of one sheet $\langle X, X \rangle = k^2$ corresponds to a “line” in the Weierstrass Model [1].

“Lines” are thus in one-to-one correspondence with the vectors l on $\langle X, X \rangle = k^2$. We shall now refer to a “line” by either giving its equation $\langle X, l \rangle = 0$ or by just giving its vector l .

A point X “lies on” the line if and only if $\langle X, l \rangle = 0$.

Given three distinct points A , B and C on the line, we will say that C “lies between” A and B if, when the line is traversed in either direction, the three points are encountered in one of the orders ACB or BCA .

Now we define a distance on H^2 .

The Lobachevskian distance between two points P and Q of H^2 is the unique non-negative number $d = d_{PQ}$ satisfying

$$\cosh \frac{d}{k} = -\frac{1}{k^2} \langle P, Q \rangle \quad (3.2)$$

With distance thus defined, we can look at the concept of an *equidistant curve*. We saw that a “line” in the Weierstrass model is defined by the section of H^2 cut by a plane passing through origin. *Equidistant curves* to one such line are defined as sections of H^2 cut by planes that are parallel to the defining plane of the line. Note that the term “parallel” is used in the Euclidean sense here.

This completes discussion of the concepts of the Weierstrass model that are relevant to the present work. The inquisitive reader may refer to [1] for a detailed treatment of the model.

3.2 The Poincaré Disk Model

This model was formulated by the French mathematician Henri Poincaré. It is an example of a finite model for hyperbolic geometry and is conformal in nature. This means that angles are represented faithfully in this model. However, distance is distorted. We will now look at how this model interprets the primitive terms.

A “point” in the Poincaré Model is defined to be a point $X = (x, y)$ in a Euclidean 2-space (R^2), such that $x^2 + y^2 < 1$. Such points comprise the interior of a circle in R^2 with center at the origin and unit radius. Note that the points on the circumference are not included in the model. We shall refer to this circle as D or “The Poincaré Disk” interchangeably.

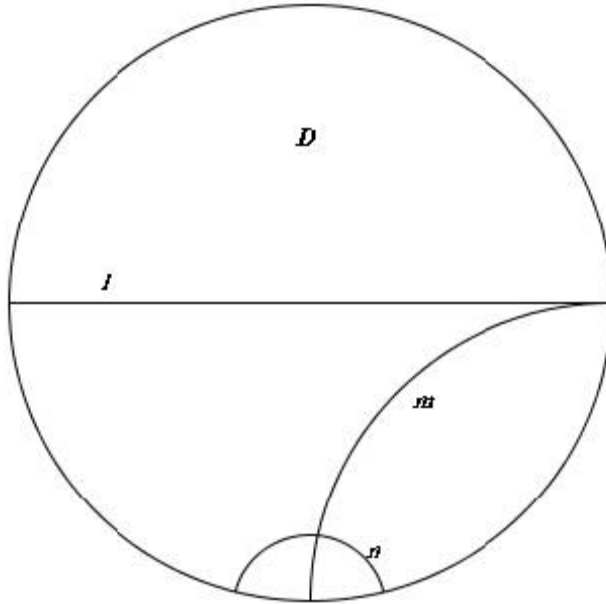


Figure 3.3 : The Poincaré model, showing lines l , m and n . We say l and n are *divergently parallel*, m and n are intersecting and l and m are *asymptotically parallel*.

A “line” is defined to be a set of interior points of D that constitute an arc of a circle that is orthogonal to D (Fig 3.3). Open diameters of D are also defined as “lines” (open diameters can be thought of as arcs of a circle of infinite radius and obviously intersect D orthogonally).

The term “lies on” and “between” have the same meaning as in the Euclidean case in this model.

Since the Poincaré Model is conformal, the angle between two intersecting “lines” can be measured by measuring the angle between the tangents to the “lines” at the point of intersection.

3.3 Isomorphism between the Poincaré and Weierstrass

Models

Quoting from [1],

An isomorphism between two models of a given set of postulates is a one-to-one correspondence between the elements, relations and operations (respectively) of one model and those of the other, such that whenever a given relationship holds among certain elements in one model, the corresponding relationship holds among the corresponding element in the other [1].

In the present work we use the Poincaré Model for a visual display of the results and do all the computations for the transformations using the Weierstrass Model. The following is a discussion of various aspects of the

isomorphism between the Weierstrass Model and the Poincaré Model that are relevant in this regard.

Consider the upper sheet of the hyperboloid $\langle X, X \rangle = -k^2$. Without loss of generality we can take the value of k^2 to be 1. Now we have the upper sheet of the hyperboloid $\langle X, X \rangle = -1$, which intersects the z -axis at the point $(0, 0, 1)$. If we project a point on this sheet down stereographically toward the point $(0, 0, -1)$, the projected point lies in a circle of unit radius with its center at the origin, in the xy -plane. This is the Poincaré Disk that corresponds to the upper hyperboloid sheet (Fig 3.4).

The projection from the Weierstrass Model to the Poincaré model is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \frac{1}{1+z} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad (3.4)$$

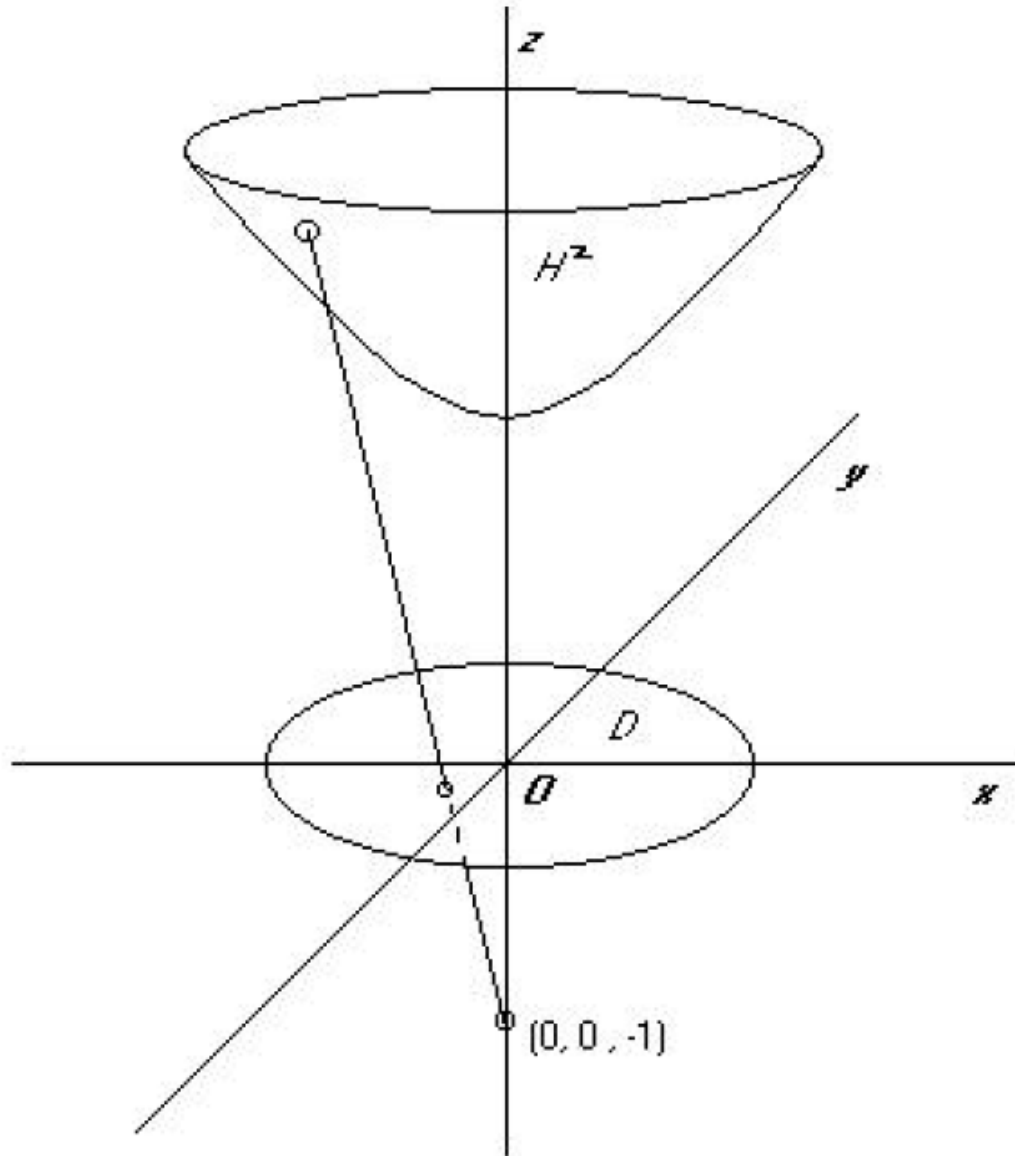


Figure 3.4 : Stereographic projection between the Weierstrass model (H^2) and the Poincaré Model (D)

And the inverse projection is given by:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \rightarrow \frac{1}{1-x^2-y^2} \begin{bmatrix} 2x \\ 2y \\ 1+x^2+y^2 \end{bmatrix} \quad (3.5)$$

3.4 The Klein Model

In geometry, the Klein Model, also called the projective model, the Beltrami-Klein model, the Klein-Beltrami model and the Cayley-Klein model, is a model of n-dimensional hyperbolic geometry in which the points of the geometry are in an n-dimensional disk, or ball, and the lines of the geometry are line segments contained in the disk; that is, with endpoints on the boundary of the disk. Along with the Poincaré half-plane model and the Poincaré disk model, it was first proposed by Eugenio Beltrami who used these models to show hyperbolic geometry was equiconsistent with Euclidean geometry. The distance function was first originated by Arthur Cayley and interpreted geometrically in hyperbolic geometry by Felix Klein.

3.4.1 Relation to the Poincaré disk model

Both the Poincaré disk model and the Klein model are models of hyperbolic space on the unit n -disk. If u is a vector of norm less than one representing a point of the Poincaré disk model, then the corresponding point of the Klein model is given by,

$$s = \frac{2u}{1 + u \cdot u}.$$

Conversely, from a vector s of norm less than one representing a point of the Klein model, the corresponding point of the Poincaré disk model is given by

$$u = \frac{s}{1 + \sqrt{1 - s \cdot s}} = \frac{(1 - \sqrt{1 - s \cdot s})s}{s \cdot s}.$$

Given two points on the boundary of the unit disk, which are called ideal points, the Klein model line is the chord between them, and the corresponding Poincaré model line is a circular arc on the two-dimensional subspace generated by the two boundary point vectors, orthogonal to the boundary of the disk. The relationship between the two is simply a projection from the center of the disk; a ray from the center passing through

a point of one model line passes through the corresponding point of the other model line.

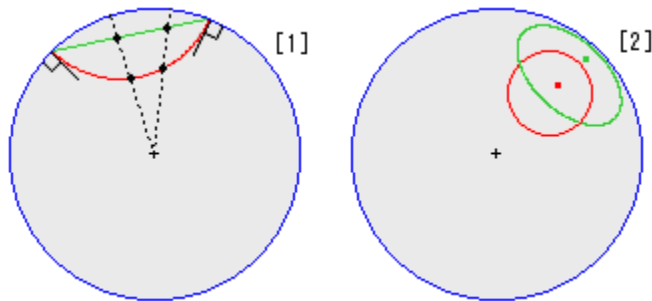


Fig 3.5 : Relation between the Klein Model and the Poincaré Model.

Both types of disks, Poincaré's and Klein's, are superimposed on each other. On Poincaré's disk model, a straight line looks like a circular arc (red) that is perpendicular to the disk edge (circumference at infinity). But on Klein's disk, a straight line is straight (green).

[1]: The red and green curves are the same hyperbolic straight line. The points on these lines correspond to each other as shown.

[2]: Hyperbolic circles have a hyperbolic radius 1. Their centers are located at a hyperbolic distance of 1.5 from the origin. A circle is drawn as a circle

(red) on Poincaré disk, but as an ellipse (green) on Klein's disk. As we can see, both centers are not at the visual centers of the circle or the ellipse.

Chapter 4

Tessellations and Hyperbolic Patterns

4.1 Tessellations

A *tessellation* is a pattern formed by arranging a basic sub-pattern in a mosaic fashion. The word “tessellate” is derived from the Ionic version of Greek word “tesseres”, which in English means “four”. The first tilings were made from square tiles. A regular tessellation is made up of congruent regular polygons. An example of a tessellation is shown below.

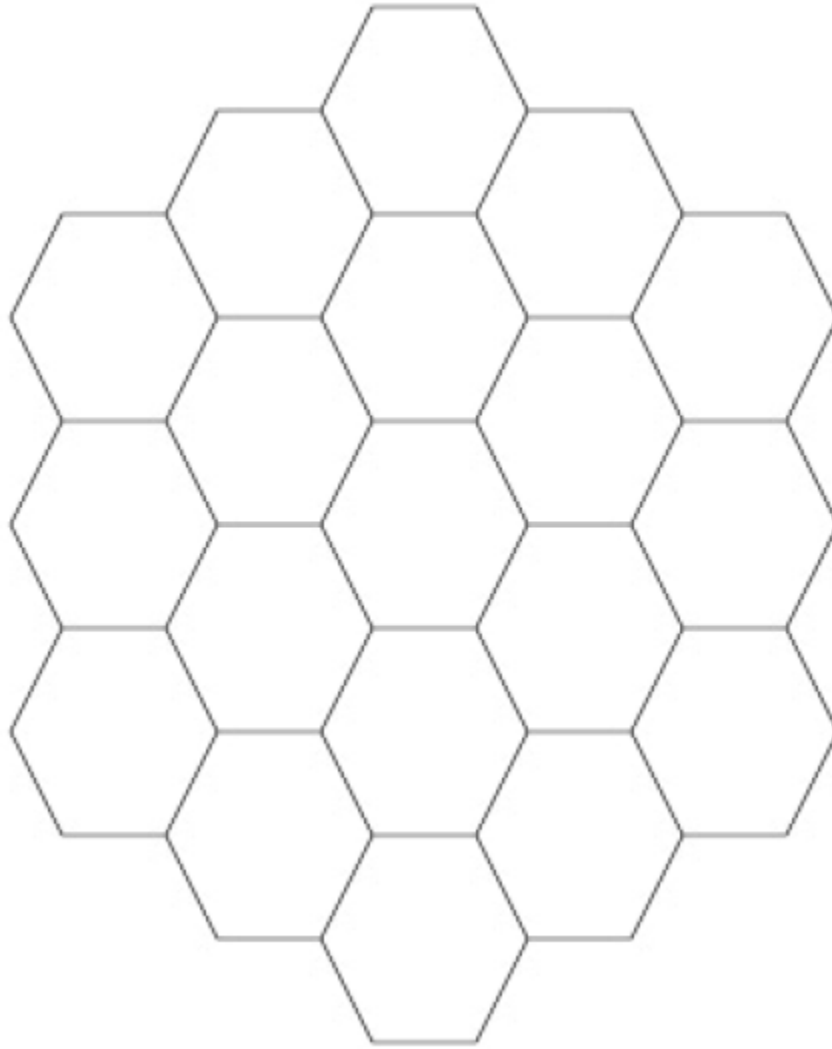


Fig 4.1 : A Euclidean tessellation made using hexagons

4.2 Repeating Hyperbolic Patterns

A repeating pattern is formed by replicating a basic sub-pattern called a *motif*.

Examples of a repeating hyperbolic pattern are the regular tessellations $\{p, q\}$ of the hyperbolic plane. The notation $\{p, q\}$ is used to denote a pattern which is comprised of regular p -sided polygons, q congruent copies of which meet at a vertex. The tessellation $\{6, 4\}$ is shown in Fig 4.2. The polygon can be irregular also. But in this document, $\{p, q\}$ is used to denote a regular tessellation of the hyperbolic plane.

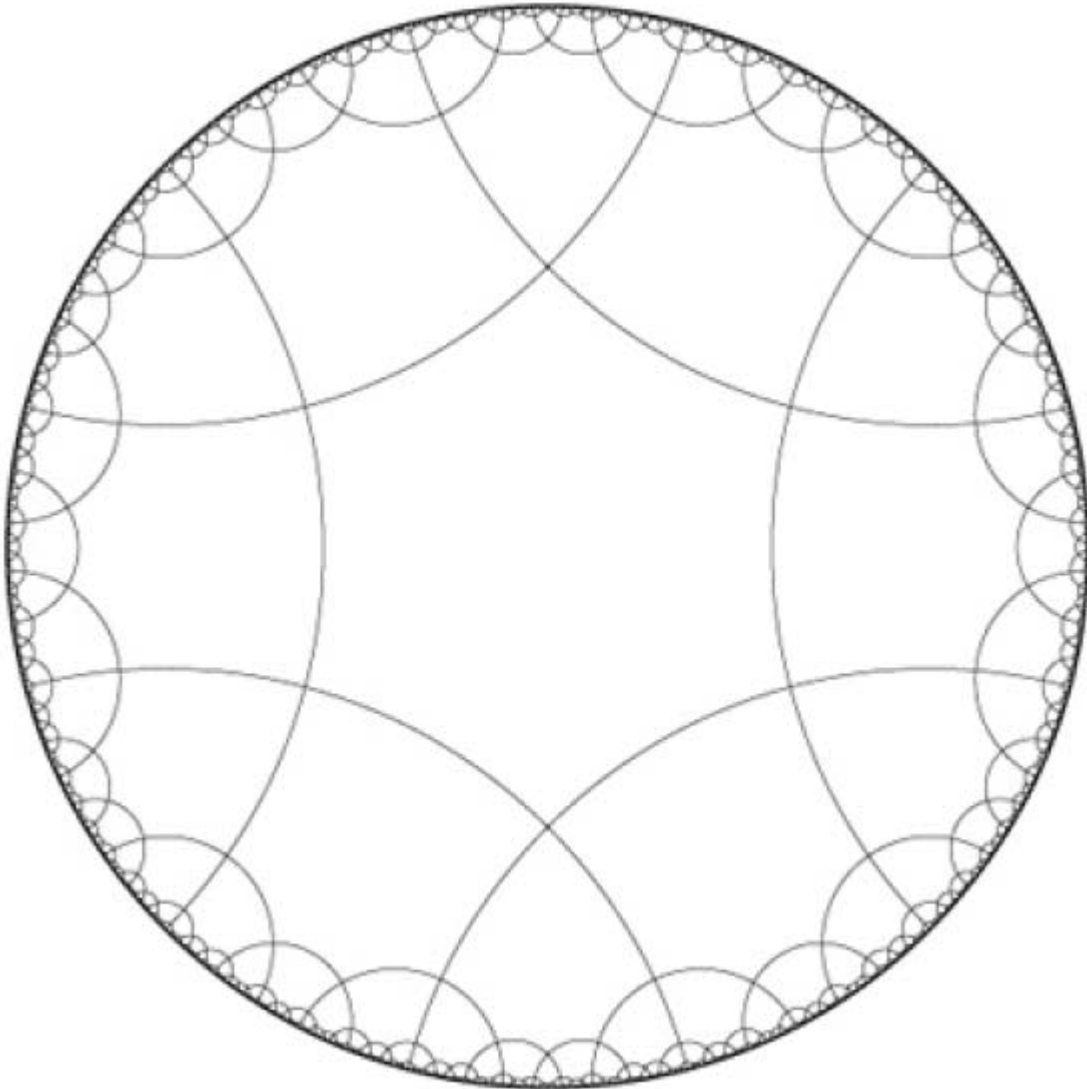


Fig 4.2 : A $\{6,4\}$ tessellation.

Also, it is necessary that $(p-2)(q-2) > 4$ to obtain tessellation of the hyperbolic plane. Tessellations with $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$ are Euclidean tessellation and spherical tessellations respectively. An example of a repeating hyperbolic pattern is shown in the figure below.

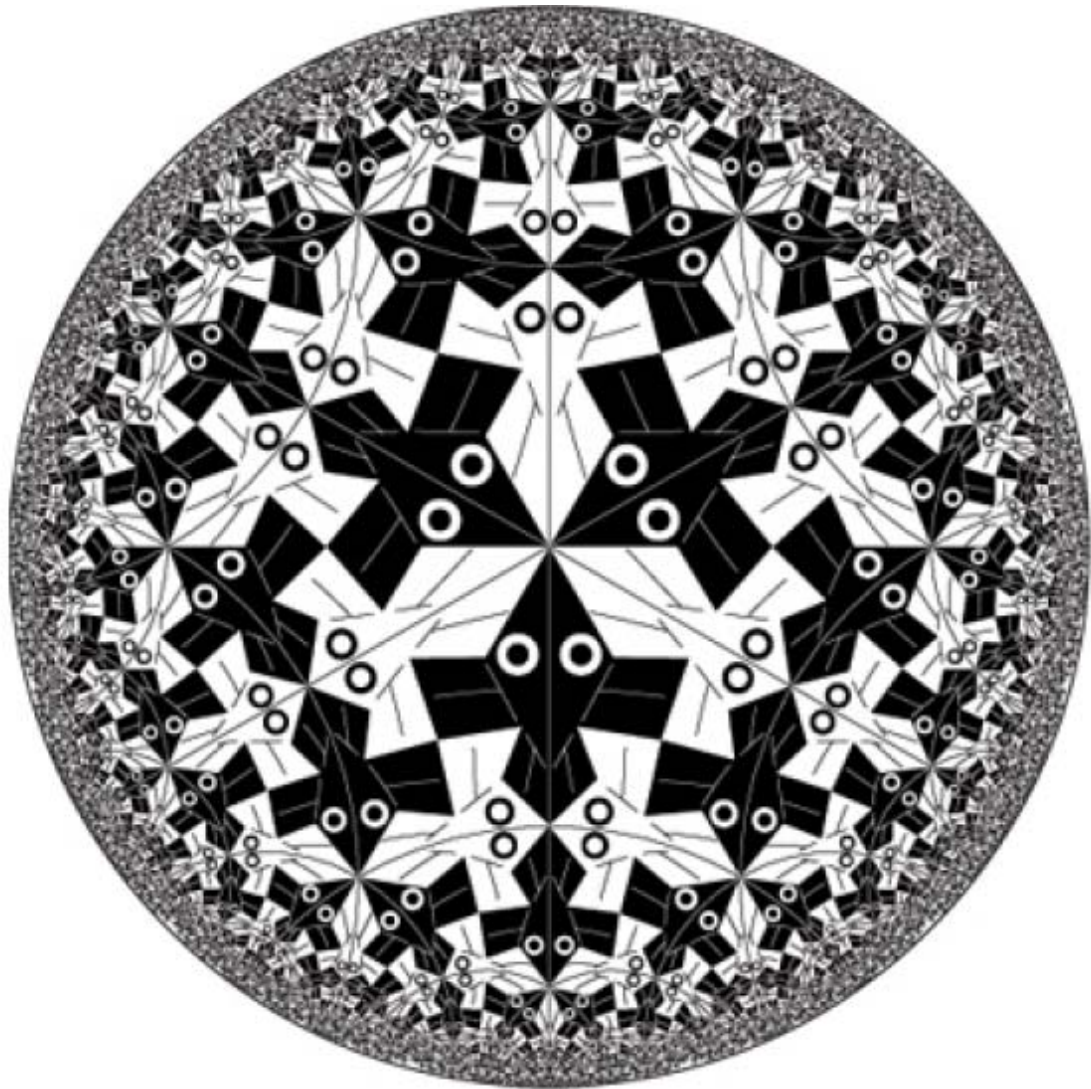


Fig 4.3 : A repeating hyperbolic pattern. It is a computer generated version of Escher's Circle Limit 1 pattern.

Chapter 5

User Interface

5.1 Description

The program is designed to allow the user to create a repeating pattern of the hyperbolic plane whose motif (or basic subpattern) is contained in one of the polygons that make up the regular hyperbolic tessellation $\{p,q\}$ by p -sided polygons meeting q at a vertex. Note that $(p-2)(q-2)$ must be greater than 4 for the tessellation to be hyperbolic.

5.2 Using the Program

The program needs a data file on which to operate. The name of the data file can be entered on the command line (e.g.: `% design cl1.dat`) or by clicking on the "Load" button at the top of the command area, which pops up a file selection dialog, which you can then use to select the desired data file. A brief description of the format of the data file is contained in the file `DATA_FORMAT`.

Depending on the data file, you may see a "framework" of hyperbolic triangles in the center of the bounding circle. One or more of these triangles make up the fundamental region of the symmetry group. To get rid of the "framework" (which is for guidelines only -- not part of the pattern), click on the "Replicate" button --> a dialog will pop up asking you if you want a framework (click on "No"); Replicate will replicate out to the currently selected number of layers (see the "Layers" array below the command menu).

The "Move To" and "Draw To" are used to (inconveniently) draw Euclidean polylines in the motif. Clicking on "Move To" allows you to start a new polyline at any place in the bounding circle where you click. Click on "Draw To" and then on a point in the bounding circle to draw a line segment from the "Move To" point. Continue clicking "Draw To" then points in the bounding circle to extend the polyline. The "Polyline" option (below) is a more convenient way of drawing polylines.

To draw a "Circle" or a "Filled Circle" click down on its center and then, holding the mouse button down, drag out a radius. When you release the mouse button, the circles will be drawn.

"Filled Poly" creates a filled (Euclidean) polygon. Clicking the left mouse button down establishes the first point; holding the button down allows you to "drag out" the first line segment; releasing the button establishes that line segment. Repeated down-clicks, drags, and releases establish additional segments. Click on the middle button to end the polygon -- this automatically closes up the polygon.

"Filled pgon" works the same way as "Filled Poly" except that it creates a polygon whose edges are (approximations to) hyperbolic line segments.

"Polyline" works the same way as "Filled Poly" and "Filled pgon" except that it creates a Euclidean polyline (i.e. not filled and not necessarily closed).

"Hyperline" creates a single hyperbolic line segment (the program has no hyperbolic polylines yet). "Hyperline (inf)" creates a single hyperbolic line

segment having its endpoints at infinity. "Hyperline (ray)" creates a single hyperbolic ray having its endpoint at infinity. Click down on one endpoint, hold the button down and drag to the other endpoint; releasing the mouse button establishes the hyperbolic line.

"Divided Line" creates a Euclidean line segment that is divided into a number of subsegments. The number of subsegments is specified in a dialog that pops up. As with a "Hyperline", click down on one endpoint, hold the button down and drag to the other endpoint; releasing the mouse button establishes the line.

Note that the original line segment is straight, but transformed copies of it may be bent (into the number of specified subsegments).

"Euclid. Line", "Equidistant", and "Perpendicular" are just for drawing "guidelines" (that can be "traced" by drawing over them with other primitives) and are not part of the motif, and go away after a "Replicate".

"Euclid. Line" just requires its two endpoints. "Equidistant" and

"Perpendicular" require 3 points to be specified by the mouse: the first two point establish a hyperbolic line, the third point establishes a distance from it in the "Equidistant" case, and it establishes a point through which the unique hyperbolic line passes that is perpendicular to the first hyperbolic line.

"Erase" simply erases the last primitive in the motif file – so it isn't the most convenient to use.

"Zoom" works -- just click on it a drag out a square -- the zoomed area ("Normal" returns to the original picture).

The "Convert" feature allows user to "warp" a motif so that it works with new values of p and q . After clicking on the "Convert" button, a dialog pops up asking for the new values of p and q . After these are entered, the program draws a garbled pattern (this is a bug). To see the correct pattern, (1) click the "Save" button at the top of the command menu, which pops up a file specification dialog, and save the warped motif to a new file, then (2) click the "Load" button and re-load the new file. Then the pattern will be more correct.

Clicking on "Print" brings up a print dialog; clicking on the Print button in the dialog sends the "Escher.ps" file to your default printer (you can name a different printer by filling in its name in the "printer name" text area).

The program currently supports 10 colors, number 1 through 10.

Clicking on one of the "Layers" buttons will replicate the pattern to that many layers.

Chapter 6

Points at Infinity

6.1 Research Problem

In the past, to use the Poincaré model, we started with a Euclidean circle and points inside the circle. In this thesis we check for points on the circle i.e. points at infinity. They are not true points of the model. They represent directions at infinity. We use them to make some constructions in the model.

6.2 Drawing lines and rays.

We consider two cases: to draw lines and rays in hyperbolic geometry in the Poincaré disk Model which have endpoints at infinity.

Case A: Line having endpoints at infinity i.e. when endpoints are anywhere on the circle whose equation is $x^2 + y^2 = 1$.

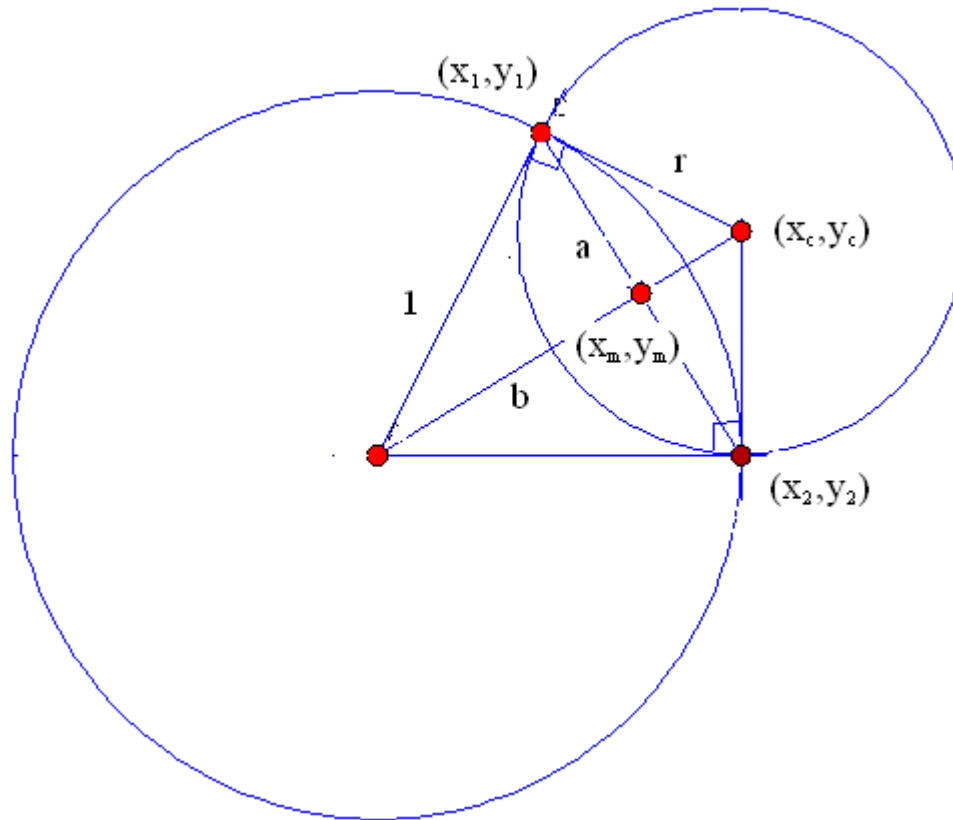


Fig 6.1 : Endpoints of the Line “Segment” are at Infinity.

From Fig 6.1 we see that (x_m, y_m) is the mid point of the line connecting two points at infinity.

Therefore, we can say that

$$x_m = \frac{x_1 + x_2}{2}$$

$$y_m = \frac{y_1 + y_2}{2}$$

Also, we see that,

$$\frac{r}{1} = \frac{a}{b} \quad \therefore \frac{r}{a} = \frac{1}{b}$$

and

$$a = \frac{1}{2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$b = \frac{1}{2} \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

From the above two equations we can write,

$$\begin{aligned} a^2 + b^2 &= \frac{1}{2} \left(x_1^2 + x_2^2 + y_1^2 + y_2^2 \right) \\ &= 1 \end{aligned}$$

This is true because, the circle is a unit circle.

Also,

$$\frac{x_c}{x_m} = \frac{r}{a} = \frac{1}{b}$$

$$\therefore x_c = \frac{x_m}{b} = \frac{x_1 + x_2}{2(x_1x_2 + y_1y_2)}$$

$$y_c = \frac{y_1 + y_2}{2(x_1x_2 + y_1y_2)}$$

Case B: We now consider a ray which has its endpoint at infinity.

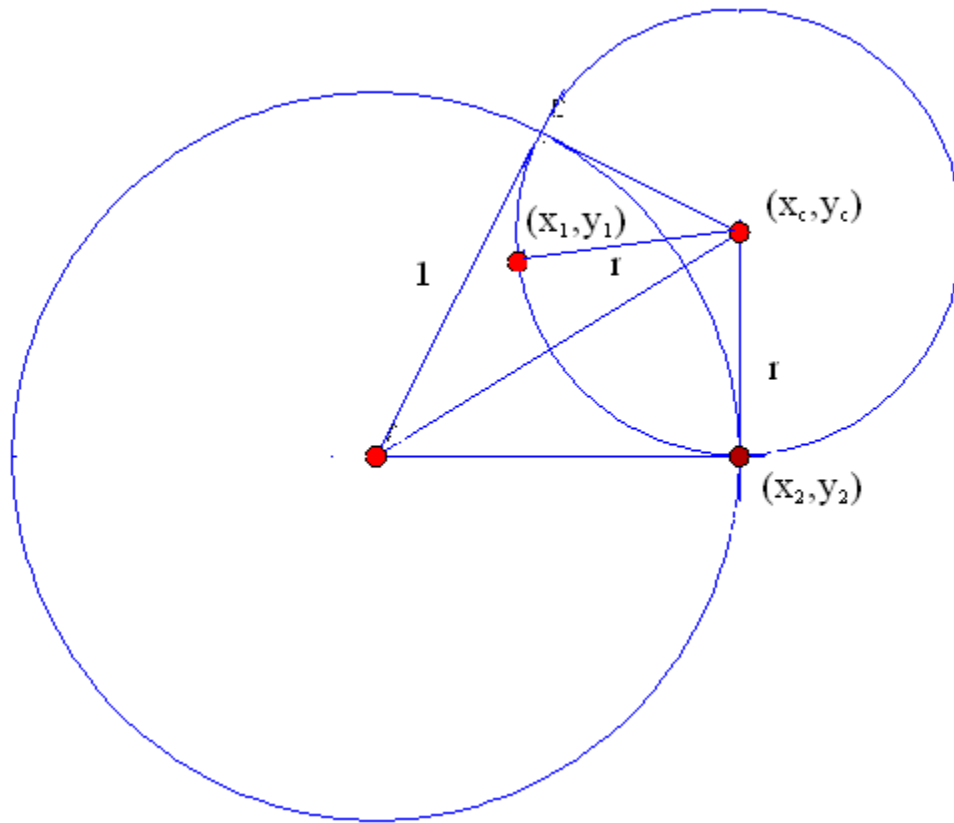


Fig 6.2. Ray with Endpoint at Infinity.

We can write

$$\begin{aligned}
 (x_c - x_1)^2 + (y_c - y_1)^2 &= (x_c - x_2)^2 + (y_c - y_2)^2 \\
 \Rightarrow x_c (x_2 - x_1) + y_c (y_2 - y_1) &= \frac{1}{2} (1 - x_1^2 - y_1^2)
 \end{aligned}
 \tag{I}$$

Also,

$$(x_c - x_2, y_c - y_2) \cdot (x_2, y_2) = 0$$

$$\Rightarrow x_c x_2 + y_c y_2 = 1$$

----- (II)

(II) - (I) gives,

$$x_c x_1 + y_c y_1 = \frac{1}{2} (1 + x_1^2 + y_1^2)$$

From

$$x_c x_2 + y_c y_2 = 1$$

we get

$$x_c (x_2 y_1 - x_1 y_2) = y_1 - y_2 \left(\frac{1}{2} (1 + x_1^2 + y_1^2) \right)$$

$$\Rightarrow x_c = \frac{1}{(x_1 y_2 - x_2 y_1)} \left[y_2 \left(\frac{1}{2} (1 + x_1^2 + y_1^2) \right) - y_1 \right]$$

Similarly,

$$y_c = \frac{1}{(x_1 y_2 - x_2 y_1)} \left[x_1 - x_2 \left(\frac{1}{2} (1 + x_1^2 + y_1^2) \right) \right]$$

From Case 1 and Case 2 of our calculations, we find the center of the circle of which the arc is a part.

Chapter 7

Results

Using the calculations from the previous chapter, we draw lines and rays on the Poincaré disk model.

Result 1:

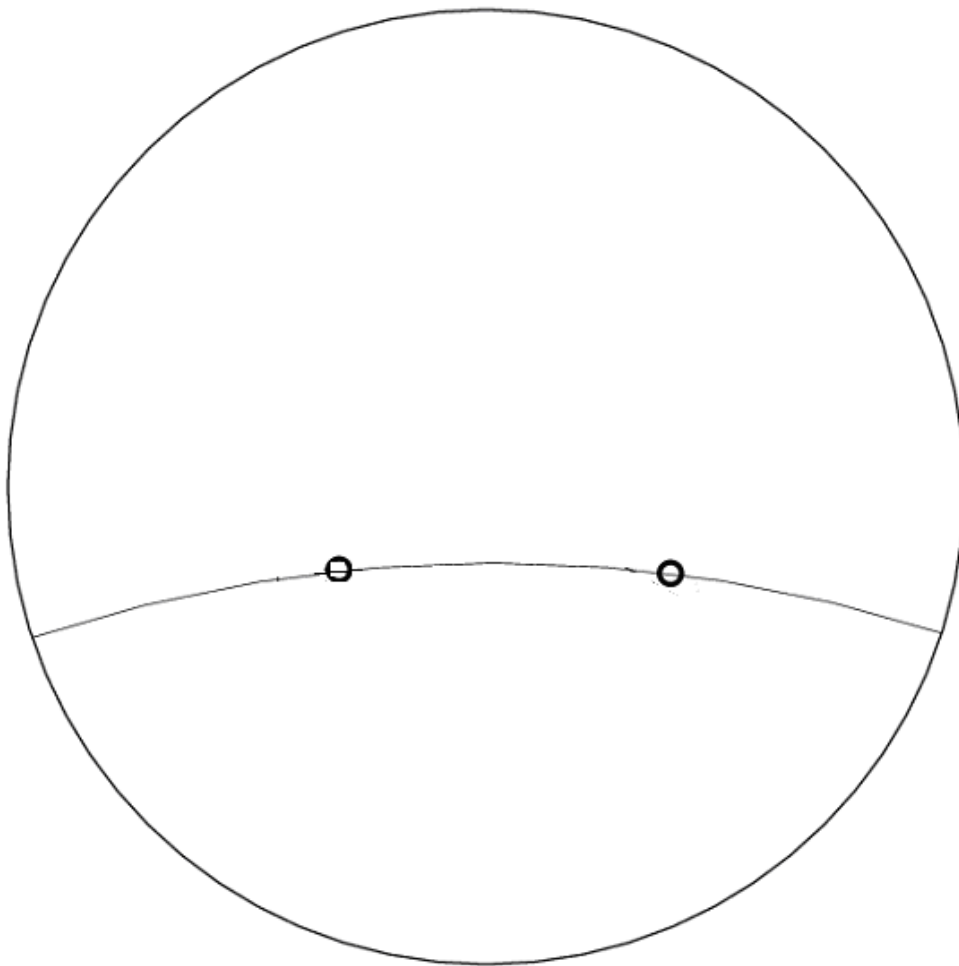


Fig 7.1 : Line Segment with its Endpoints at Infinity.

Result 2:

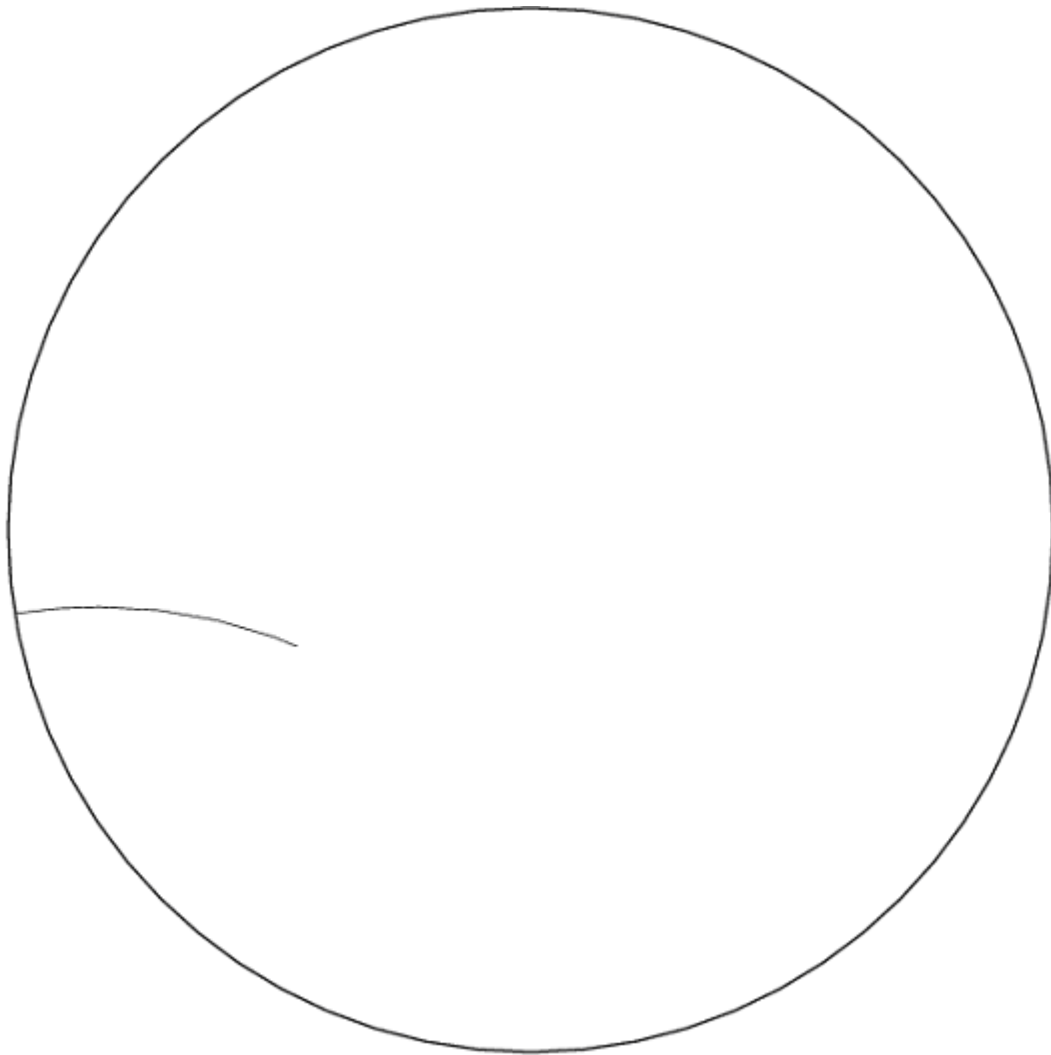


Fig 7.2 : Ray with Endpoint at Infinity.

In the following figures we see the output with repeating patterns.

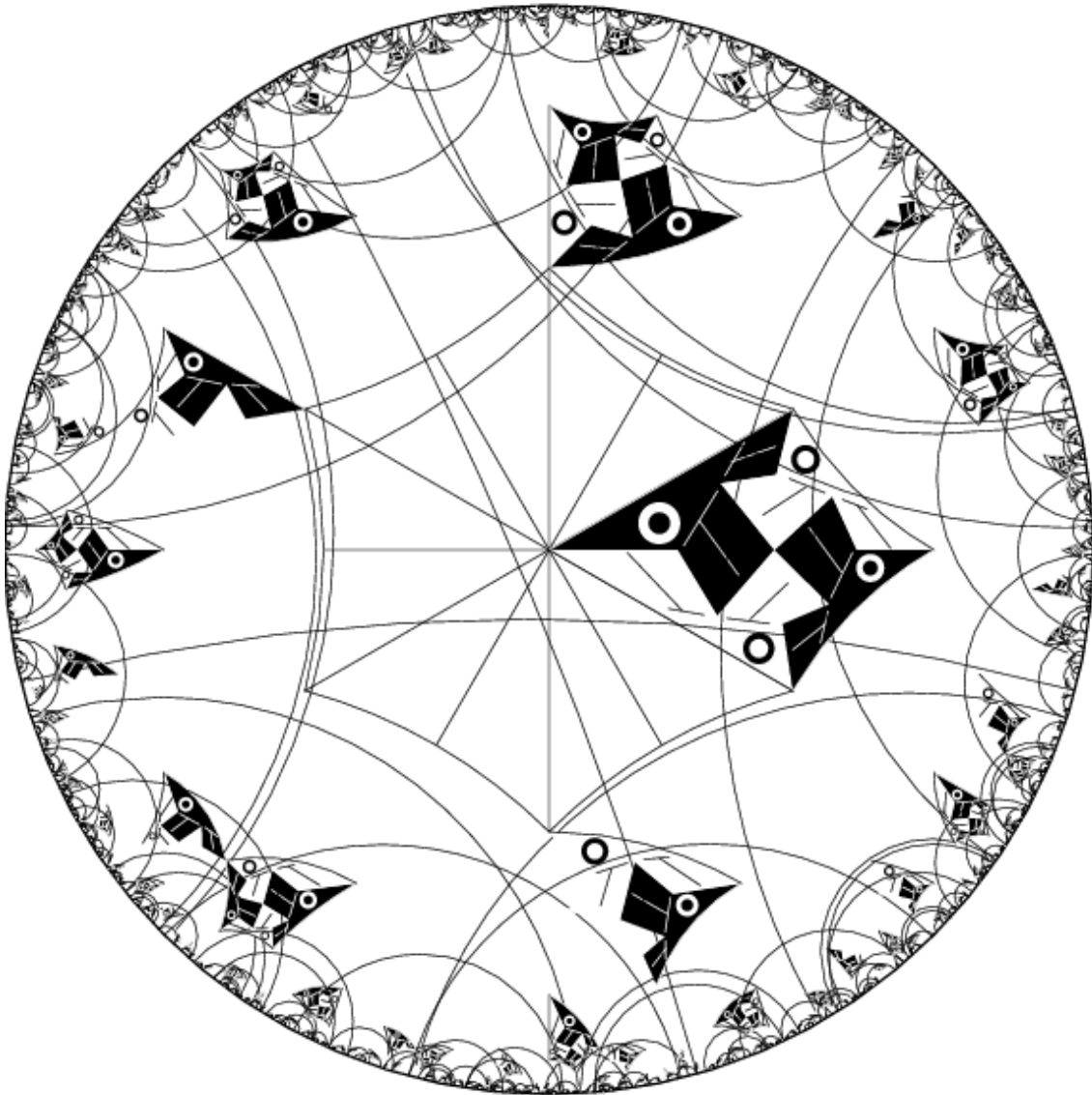


Fig 7.3 : Repeating Patterns (Line Segment with Endpoints at Infinity).

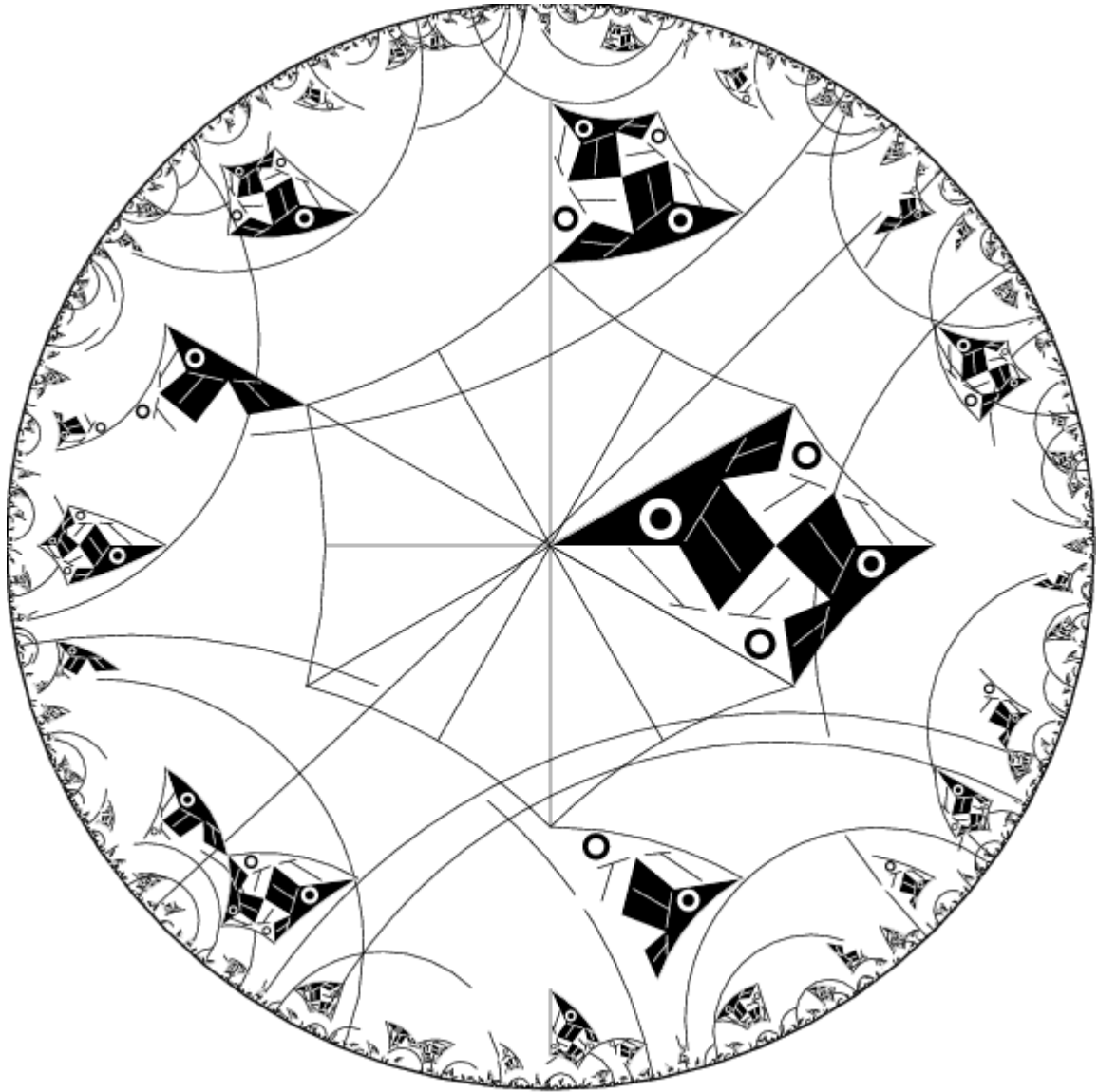


Fig 7.4 : Repeating Patterns (Ray with Endpoint at Infinity).



Fig 7.5 : Repeating Patterns (Line Segment with none of its Endpoints at Infinity).

CONCLUSION

In this research we designed and implemented methods to draw hyperbolic lines and rays having endpoints at infinity. The Poincaré model of hyperbolic geometry was used for representation of the patterns.

We tested the method by considering various points at infinity. The results obtained were correct. The user interface, which was a part of the design program, was also enhanced to handle the lines having endpoints at infinity. This research is a contribution to Dr. Dunham's research on points at infinity.

Future Work

We propose the integration this combination of data structures and algorithm that draws complete hyperbolic lines and hyperbolic rays with Tarun Kapoor's [3] work which draws repeating patterns based on the regular tessellation $\{p, \infty\}$ and $\{\infty, q\}$.

Appendix 1

Data Format

Here we explain the format of the data file that the hyperbolic “design” pattern uses. In most cases, it is not necessary to understand how it works. Unfortunately, the design program is not at all robust, so that any error in a data file will probably cause the program to crash.

This program can only create hyperbolic patterns that are based on the regular tessellations (p, q) of regular p -sided polygons meeting q at a vertex ($(p-2), (q-2)$ must be > 4 to be hyperbolic). For simplicity, we will call a regular p -sided polygon a “pgon” and we always arrange that one of them, the “central pgon” is centered within the bounding circle.

Here is a sample data file, “cl2.dat” that creates Escher’s Circle Limit II pattern:

```
8 3 2 0 3 1
1 2 3
1 2 3
2 2 3 1
```

1 3 1 2
 2 2 3 1
 1 3 1 2
 2 2 3 1
 1 3 1 2
 2 2 3 1
 1 3 1 2
 20
 0.000000e+00 0.000000e+00 1 4 6
 1.368406e-01 1.379250e-01 1 5 6
 2.575625e-01 2.575625e-01 1 5 6
 2.575625e-01 1.893719e-01 1 5 6
 1.837032e-01 1.160219e-01 1 5 6
 1.215594e-01 5.387817e-02 1 5 6
 6.756561e-02 0.000000e+00 1 5 6
 0.000000e+00 0.000000e+00 1 6 6
 6.756561e-02 0.000000e+00 2 4 6
 1.215594e-01 5.387817e-02 2 5 6
 1.837032e-01 1.160219e-01 2 5 6
 2.575625e-01 1.893719e-01 2 5 6
 2.575625e-01 2.575625e-01 2 5 6
 3.747406907514156e-01 1.552226764822982e-01 2 5 6
 2.188499e-01 0.000000e+00 2 5 6
 6.756561e-02 0.000000e+00 2 6 6
 2.188499e-01 0.000000e+00 3 4 6
 3.747406907514156e-01 1.552226764822982e-01 3 5 6
 3.645668590273164e-01 0.0000000000000000e+00 3 5 6
 2.188499e-01 0.000000e+00 3 6 6

In the first line, 8 3 2 0 3 1

The first number is the value of p , i.e. $p = 8$ in this case.

The second number is the value of q , i.e. $q = 3$ in this case (thus this pattern is based on the tessellation (8.3)).

The third number, 2 in this case, is the number of “different” sides of the central p-sided polygon that are used to form the fundamental region that contains the motif (the other sides of the fundamental region are two radii from the center to two vertices of that p-sided polygon separated by $2*(2*\pi/p)$). This number must divide p, and p divided by this number is the number of copies of the motif that appears in the central p-sided polygon.

The fourth number, 0 is not used (it is there to maintain compatibility with older versions of this program).

The fifth number, 3 in this case, must be the highest “color” number of the colors used. The colors are numbered starting at 1 (black), 2(white), 3(red), 4(green), and so on.

The sixth number, 1 in this case, indicates the kind of reflection symmetry the pattern has within the central p-sided polygon:

0 indicates that there is no reflection symmetry (only rotation symmetry).

1 indicates that there is reflection symmetry across the perpendicular bisector of one of the edges of the p-sided polygon.

2 indicates that there is reflection symmetry across a radius (from the center to a vertex of the p-sided polygon).

The second line, 1 2 3, is the color permutation induced by rotating by $2 \cdot (2 \cdot \pi / p)$ (i.e., the third number of line 1 times $2 \cdot (2 \cdot \pi / p)$). Note that this is the “array” representation of permutations (not the “mathematical” one using cycles): the values listed are the values of `perm[1]`, `perm[2]`, `perm[3]`, etc., so 1 2 3 represents the identity permutation in this case.

The third line, 1 2 3, is the color permutation induced by the reflection if the sixth number of line 1 is 1 or 2 (it is just the identity if the sixth number is 0). In this case, 1 2 3 is the identity permutation.

The next p lines consist of a first number followed by a color permutation.

The first number of the first line indicates which edge (edge 2 in this case) of the transformed pgon should lie next to edge 1 of the central pgon. In general, if this first number is positive, the transformed pgon is rotated into position; if the number is negative, a reflection is used to move the transformed pgon into position. Note that the edges are numbered from 1 to p , not from 0 to $p-1$.

The color permutation of the first line, 2 3 1 (i.e. `perm[1] = 2`, `perm[2] = 3`, `perm[3] = 1`) is non-trivial in this case (i.e. black \rightarrow white \rightarrow red \rightarrow black)

The first number of the second line indicates which edge (edge 1 in this case) of the transformed pgon should lie next to edge 2 of the central pgon.

In this case, the color permutation is 3 1 2.

The first number of the third line indicates which edge (edge 2 in this case) of the transformed pgon should lie next to edge 3 of the central pgon. In this case the color permutation is 2 3 1.

The first number of the fourth line indicates which edge (edge 1 in this case) of the transformed pgon should lie next to edge 4 of the central pgon. In this case, the color permutation is 3 1 2.

.
. .
.

The first number of the eighth line indicates which edge (edge 1 in this case) of the transformed pgon should lie next to edge 8 of the central pgon. In this case, the color permutation is 3 1 2.

The next line consists of a single number, the number of points that make up the motif. It is 20 in this case.

Following that are 20 lines of 5 numbers each; each line specifies one point.

Each line has the following format:

x-coordinate y-coordinate color type-of-point number-of-layers

where:

the x- and y-coordinates are within the central pgon (and hence within the unit circle).

the color is one of 1, 2, ... , 10

the type-of-point is one of:

1 -- a "Move To"

2 -- a "Draw To"

3 -- a "Circle" (there should be two of these in succession)

4 -- start a (Euclidean) "Filled Poly"

5 -- continue a (Euclidean) "Filled Poly"

6 -- end a (Euclidean) "Filled Poly"

7 -- a "Hyperline" (there should be two of these in succession)

8 -- a "Filled Circle" (there should be two of these in succession)

9 -- start a (Euclidean) "Polyline"

10 -- continue a (Euclidean) "Polyline"

11 -- end a (Euclidean) "Polyline"

12 -- start a (hyperbolic) "Filled pgon"

13 -- continue a (hyperbolic) "Filled pgon"

14 -- end a (hyperbolic) "Filled pgon"

the number-of-layers is not used (is there for compatibility with previous versions of the program).

References

1. Faber Richard L. (1983). Foundations of Euclidean and Non-Euclidean Geometry. Marsel Dekker, Inc, New York and Basel.
2. Greenberg M. J. (1994). Euclidean and Non–Euclidean Geometries. W.H. Freeman and Company, New York.
3. Kapoor T. K. (2005). Generating Repeating Hyperbolic Patterns based on $\{p, \infty\}$.
4. Trudeau Richard J. (1987). The Non-Euclidean Revolution. Birkhäuser, Boston.