Development of a Vegetation-Soil-Consumer Model with Harvesting

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University of Minnesota Duluth

Submitted by Tom Sjoberg

Advisor Dr. Harlan Stech
Department of Mathematics and Statistics
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Abstract:

Through the derivation, parameterization, and simulation of a differential equation model, we will analyze the relational impacts in the vegetation-soil-consumer ecosystem. There are numerous parameters that affect this ecosystem. However, we will focus on the impact of harvesting the vegetation, or harvesting the consumer, or the introduction of fertilization to the soil to determine the impact that these activities have on the stability of the ecosystem. We will find that there are parameter values that allow the system to be at a favorable stable equilibrium, and there parameter values that will create an unstable ecosystem, or cause the system to crash to an unfavorable state.
1. **Introduction:**

Vegetation density and soil erosion are factors that significantly affect watershed ecosystems. There are numerous considerations in the triad relationship between consumer-vegetation-soil ecosystems. Soil conditions influence vegetation growth, which (through litter decay) increases soil depth. Consumer biomass levels are affected by the vegetation on which they feed, and they regulate vegetation biomass levels. Moreover, harvesting and the processes represented by multiple parameters affect these ecosystems. In recent years, mathematical modeling has been used to demonstrate the dynamical processes in the consumer-vegetation-soil ecosystem. Many efforts have been made to model vegetation dynamics under ecological stresses. The development of differential equations to express the dynamical process of the ecosystem is a common approach in the modeling method.

In one of the attempts to model this system, Johan van de Koppel and Max Rietkerk (2000) studied the mathematical implications of vegetation-regulated herbivore population dynamics. They concluded that herbivore populations impact the standing crop of vegetation and irreversible vegetation change may occur through overgrazing. Their use of phase plane analysis was helpful to their understanding of the dynamics of a plant-herbivore system. The phase plane gave a graphical interpretation of isoclines (nullclines) between the vegetation and consumer with equilibria at the points of intersection. Their model supported stable and unstable equilibria dependent on the conditions in the ecosystem. Thus, they anticipated different results dependent on the soil conditions.

A vegetation-erosion model was created by: Z.H. Wang, G.H. Huang, G.Q. Wang, and J. Gao (2004). The model was applied to the Xiaojiang, Heishui, and Shengou Watersheds, tributaries to the Yangtze River in Southern China. Their study, using a differential equation model, allowed the researchers to quantify vegetation-erosion dynamics in order to improve the sustainability of the
watersheds. However, the model they used demonstrated some limitations due to inaccurately modeling erosion processes. Their erosion process was exponential, without limitation.

Livestock-grass-soil systems were considered in the modeling of Javier Ibanez, Jaime Martinez, and Susanne Schnabel (2007). They studied desertification due to overgrazing in the dry lands of the Mediterranean Region. In their work they used the logistic growth model of populations to represent the primary production of grass. The logistic growth model is an important construction that will be used in this paper as well. It will be an important part of the differential equation representing the growth rate of vegetation.

As livestock grazed on the drylands of the Mediterranean, consumption of the vegetation was observed and recognized as having an impact on the vegetation, soil, and hydrology of the area. Parametric values were established for three extensive livestock farming systems in Spain. However, the notation for their model was somewhat awkward and it seemed the notation could be simplified.

Another study of herbivore-vegetation dynamics in a semi-arid grazing system is described in the work of Meza, Bhaya, Kaszkurewicz, and Costa (2006). They proposed a mathematical system to explain the alternative equilibrium points in a dynamical system sensitive to soil degradation. An “on-off” policy was developed to manage livestock grazing around stable equilibrium points. The objective was to manage a sustainable herbivore-vegetation ecosystem and not allow the ecosystem to receive irreversible damage. They used simulation models to graphically depict the no grazing regions and the grazing regions when livestock were allowed to forage the vegetation.

The goal of this project is to design an alternate, process-based, differential equation model that realistically describes particular vegetation-soil-consumer ecosystems. We will use phase planes to interpret boundaries to the system in section 3, and numerical simulations to demonstrate tendencies when parameter values are altered. We would like to clarify previous models representing the
consumer-vegetation-soil ecosystem and will create a special-purpose simulator used for performing numerical experiments. Simulations of the model are used in section 4 to interpret the results when consumer and vegetation harvesting are introduced. Also, we are interested in adding nutrients to the soil and determining the impacts on equilibria and ecosystem stability. In section 5, we discuss implications and suggest a few topics for further study regarding this model. The Mathematica7(2009) software program has a number of different tools for simulating systems of differential equations and graphing the solutions of two and three dimensional differential equations. In the Appendix are found the Mathematica codes used to perform the simulations. References are included in the final section where there is a distribution of work five years or older, and references to work written within the last five years.
2. Model Derivation:

There is a select group of equations and functions that allow us to mathematically describe an ecosystem and determine the ecological direction the environmental system is headed. The motivation of our model is the classic producer/consumer model of Rosenzweig and MacArthur (1963):

\[
V' = V \left[ \frac{d(1-V)}{k} - \frac{P_{\text{max}}C}{P_{\text{max}}H + V} \right]
\]

\[
C' = C \left[ \frac{\gamma P_{\text{max}}V}{P_{\text{max}}H + V} - \frac{m}{\text{mortality}} \right]
\]

In their work, they assume logistic growth of the vegetation, and a Holling Type II model of consumer harvesting of the vegetation. The Holling Type II function (1959) was first used in ecology by C. S. Holling to determine the processing rate of the consumer.

We consider a system consisting of vegetation, \( V(t) \), a consumer, \( C(t) \), and soil, \( S(t) \) as measured by their nutritional levels. The vegetation growth rate is represented by a logistic growth model which is modulated by soil nutrient conditions of the soil-vegetation ecosystem. The consumer growth rate is directly influenced by the amount and quality of vegetation. The consumer foraging rate is represented by a saturating function of vegetation quantity. Soil grows by the decay of vegetation litter and soil nutrient addition. We assume soil growth due to weathering is nil.
The following diagram describes the relationships between C, V, and S and parameters (described later) that affect them.

Figure 2.1:
A diagram showing the flow of nutrients through the system.
The change over time of the vegetation, consumer, and soil components are expressed in the foregoing differential equations:

\[
V' = V \left[ \frac{dS}{b+S} \left(1 - \frac{V}{k}\right) - \frac{P_{\text{max}} C}{P_{\text{max}}(H+V)} - L_{\text{death}} - h_{1V} \right]
\]

\[
C' = C \left[ \frac{P_{\text{max}} V}{P_{\text{max}}(H+V)} - \frac{m}{m_{\text{mortality}}} - h_{2V} \right]
\]

\[
S' = \left( F_{\text{addition}} + aL_{\text{decomposition}} \right) - \left( \frac{\alpha}{1 + \beta V} \right) S
\]

The rate of growth of vegetation over time is a product of a saturating function of the amount of soil present, multiplied by the logistic growth of the vegetation, minus consumer harvesting, minus vegetation mortality, minus managed vegetation harvesting. Vegetation growth is dependent on soil quality. The traditional logistic growth model, \( V(1 - \frac{V}{k}) \), is multiplied by a saturating function of soil levels, \( \frac{dS}{b+S} \). As \( S \to +\infty \), \( k \) is the maximum carrying capacity of the vegetation assuming high soil nutrient. The growth rate of the vegetation is reduced by the rate of vegetation harvested by the consumer. The half-saturation constant, \( HP_{\text{max}} \), affects the rate of vegetation processed. Non-foraging mortality loss of vegetation is included, as is a possible means of including a vegetation harvesting rate, \( h_{1V} \).

The growth rate per capita for the consumer is modeled as the foraging rate, times a coefficient, \( 0 \leq \gamma \leq 1 \), the conversion efficiency of vegetation into consumer biomass. There is
mortality of the consumers, $mC$, subtracted from the rate of growth, along with a possibly managed consumer harvesting rate, $h_2C$.

Soil quality (nutrient) has a growth rate dependent upon the decay of vegetation, and on the addition of nutrients to the soil, $F$. There is a loss of soil nutrient due to erosion that is assumed to be a decreasing function of vegetation cover. For simplicity, the rate of erosion of soil is assumed to take the form $S \left(\frac{a}{1+\theta V}\right)$, $a$ the asymptotic loss rate of soil coefficient as $V \rightarrow 0$.

The Wang, et.al\textsuperscript{8} paper couples vegetation and erosion rate as dynamic processes in their model of watershed systems. In our model the dynamic process is between vegetation, consumer, and soil, where we consider erosion as an intrinsic rate loss of soil reduced by the amount of vegetation. The Wang, et.al\textsuperscript{8} paper treats the erosion rate as the most important entity that affects vegetation growth. In contrast, our model considers erosion as a rate loss of soil nutrient, reducing the availability of soil nutrient to the vegetation.

Our use of the logistic growth function to represent vegetation growth is similar to the Ibanez, et.al\textsuperscript{1} study. They have the primary production of grass represented by a logistic function where (in their notation) $PP = GR \times G \left(1-{\frac{G}{GK}}\right)$. $GR$ is the intrinsic growth rate of grass and $GK$ is the carrying capacity of grass. Both are assumed proportionate to a function of the form: $1 - \exp\left[\frac{\text{Max}(S \cdot msg, 0)}{sm1}\right]$. A reduction in soil nutrient availability negatively affects vegetation growth rate and the carrying capacity of the vegetation. Vegetation growth is dependent on soil nutrient availability as is the carrying capacity of grass. The Ibanez, et.al\textsuperscript{1} paper uses an awkward computer-code notation and equation setting. Notation and equations in our paper are meant to be more easily understood. However, their parametric values were useful to our work in setting intervals for the intrinsic maximum growth rate of grass and the maximum carrying capacity of grass.
Koppel and Rietkerk\textsuperscript{2} studied the effects of herbivore regulation on semi-arid grazing systems as a means to avoid irreversible vegetation change. In their model, soil quality is a function of water availability. Their equations are similar in form to a 2-dimensional simplification of our model. Their use of phase plane representations clarified where equilibria occurred as densities of herbivore and vegetation increased. Their system was two dimensional (as we aforementioned) when we set soil quality to be a function of vegetation density. Coexistence equilibria have a spiral rotation into the equilibrium point if it is stable with complex eigenvalues and a spiraling out from the point if it is unstable with complex eigenvalues. The phase plane representations used by Koppel and Rietkerk\textsuperscript{2} were hand-drawn, but we have chosen to use a software program, Mathematica, to illustrate the phase planes.

Mesa, et.al\textsuperscript{4} continued the study of Koppel and Rietkerk using an “on-off” policy for the herbivore-vegetation dynamics in a semi-arid grazing system. It has been recognized for a considerably long time that alternative vegetation states occur in semi-arid grazing systems. Simulations of the alternate vegetation and herbivore densities resulted in their better understanding of the sudden and irreversible jumps between vegetation states.

3. Model Analysis

If we assume that the rate of soil gain/loss is small relative to that of the vegetative and consumer components, we can set $\dot{S} = 0$ in the third equation. We can let $S = \bar{S}$, where

$$\bar{S} = \frac{1}{\alpha} (1 + \beta V)(F + aLV),$$

a function of $V$. This reduced form shares the same essential form as the Koppel, et.al\textsuperscript{2} and Mesa, et.al\textsuperscript{4} models.

If we substitute $\bar{S}$ into the first equation, the model becomes two dimensional:
\[ V' = V \left( \frac{d \tilde{S}}{b + \tilde{S}_{\text{growth}}} (1 - \frac{V}{k}) - \frac{P_{\text{max}} C}{P_{\text{max}} H + V} \right) - \frac{L_{\text{death}}}{h_{\text{harvesting}}} \]

\[ C' = C \left[ \frac{P_{\text{max}} V}{P_{\text{max}} H + V} - \frac{m_{\text{mortality}}}{h_{\text{harvesting}}} - \frac{h_{\text{harvesting}}}{h_{\text{consumption}} H} \right] \]

Parameter default values used in our simulations of the differential equations for the two dimensional system are based on Ibanez\(^1\)(2007) and Meza\(^4\)(2006).

- asymptotic soil fertility coefficient, \(d=1.0\)
- half-saturation soil constant, \(b=0.20\)
- carrying capacity, \(k=0.95\)
- maximum rate of processing, \(P_{\text{max}}=1.0\)
- half-saturation determining constant for maximum rate of processing, \(H=0.10\)
- litter growth, \(L=0.40\)
- harvesting vegetation, \(h_1=0.0\)
- conversion of vegetation into consumer biomass, \(\gamma=0.88\)
- mortality, natural death of consumer, \(m=0.64\)
- harvesting consumer, \(h_2=0.0\)
- soil nutrient addition, \(F=0.0\)
- conversion of litter into soil nutrient, \(a=0.04\)
- asymptotic loss of soil coefficient, \(\alpha=0.01\)
- soil erosion sensitivity constant, \(\beta=100\)
We will use our differential equations to develop a phase plane graphical analysis. Like the Koppel and Rietkerk study, phase plane graphical analysis is a tool used to visually locate states of equilibrium in the consumer-vegetation coordinate system. The nullclines, or zero net growth isoclines, of the consumer and vegetation are lines where the states of equilibria for both entities will lie at the nullclines intersection.

Setting, \( \frac{\gamma P_{\text{max}} V}{H P_{\text{max}} + V} - m - h_2 = 0 \), we can find the nullcline for the consumer.

Specifically from,

\[ \frac{\gamma P_{\text{max}} V}{H P_{\text{max}} + V} = m + h_2, \]

we obtain the nullcline for the consumer: \( V = \frac{H P_{\text{max}} (m + h_2)}{\gamma P_{\text{max}} - (m + h_2)} \).

Setting, \( \frac{d \bar{S}}{b + \bar{S}}(1 - \frac{V}{k}) - \frac{P_{\text{max}} C}{H P_{\text{max}} + V} - L - h_1 = 0 \), we can find the nullcline for the vegetation.

That is,

\[ \frac{P_{\text{max}} C}{H P_{\text{max}} + V} = \frac{d \bar{S}}{b + \bar{S}}(1 - \frac{V}{k}) - L - h_1 \]

from which we obtain the nullcline for the vegetation:

\[ C = \frac{H P_{\text{max}} + V}{P_{\text{max}}}\left[ \frac{d \bar{S}}{b + \bar{S}}(1 - \frac{V}{k}) - L - h_1 \right] \]

with \( \bar{S} = \frac{1}{a}(1 + \beta V)(F + aLV) \).
We will use these nullclines to partition the phase plane. We can analyze the phase planes to determine system equilibrium points under variations of the parametric values. Consider the nullclines, the system always supports a “no-life” equilibrium \((V,C)=(0,0)\) and “vegetation only” equilibrium \((V,C)=(V^*,0)\) when \(V^*\) solves \[
\frac{d\bar{S}}{b + \bar{S}} (1 - \frac{V}{k}) - L - h_i = 0,
\]
with \(\bar{S} = \frac{1}{\alpha} (1 + \beta V^*)(F + \alpha LV)\). The following figures illustrate that the model can support two “vegetation only” equilibrium states. The default values are used.

![Figure 3.1a: Allee effect; \(h_i=0.35; (V_0,C_0)=(0.70,0.05); H=0.10; V'=0\), the unimodal; \(C'=0\), the vertical line](image)

The simulations illustrate that the sustainability of the system depends on the systems initial state. The “vegetation only “ model possesses an Allee effect, Stephens, et.al\(^7\). The Allee effect refers to a phenomena in biology where a population cannot sustain itself below a certain critical level. If \(V\) is less than the lower valued \(V^*\), then \(V\) tends toward zero as \(t\) tends toward positive infinity, Figure 3.1a.
Otherwise, $V$ tends toward the larger valued $V^*$, which is the carrying capacity of the consumer-free system, Figure 3.1b.

![Figure 3.1b: Allee effect; $h_1=0.35$; $(V_0,C_0)=(0.70,0.04)$; $H=0.10$; $V'=0$, the unimodal; $C'=0$, the vertical line. The lesser consumer population of $C=0.04$ versus $C=0.05$ allows the system to tend to the larger $V^*$ when there are no other changes in the system. When the two nullclines intersect in the region $C > 0$ there is also a third ecologically relevant “coexistence” equilibrium at the point of intersection. The following figures depict possible phase planes and solution trajectories for our model. Also, the figures show how model behavior is influenced by the predation half-saturation parameter, $H$.](image-url)
The default parametric values are used, and $V(0)=0.70$ and $C(0)=0.05$. We find an inward spiraling stable equilibrium point when the half-saturation determining constant is $H = 0.1$.

The same parametric values are used except we reduced the half-saturation determining constant, $H$, by .01 to .09. A periodic orbit formed by a Hopf Bifurcation is found by our initial conditions for the vegetation and consumer.

We changed $H$ to 0.08 moving the consumer nullcline to the left. The vegetation and consumer both are seen to crash to $V = 0$, $C = 0$, where vegetation and consumer have both died off. Additional simulations (not shown) indicate the system does not support periodic orbits for these parameter values.
To summarize, in the preceding Figure 3.2, we have all parametric values set at default values creating an inward spiraling coexistence equilibrium for the vegetation and consumer nutrient populations. The vertical line is the consumer nullcline where the rate of change over time of the consumer population is equal to zero. The unimodal curve is the vegetation nullcline, where the rate of change over time also is equal to zero. Figure 3.3 shows a periodic orbit equilibrium created by a slight decrease in H in the default parametric values. We reduced the half-saturation determining constant, H, by one-hundredth which changed an inward spiraling stable equilibrium into a periodic orbit. In Figure 3.4, we “crashed” the consumer and vegetation populations \((V,C)=(0,0)\) by further reducing H to 0.08.

4. Application of the Model: The Effects of Harvesting

In this section we examine through simulations the possible impact of various management practices. Specifically, we will consider vegetation and consumer harvesting, as well as the impact of fertilization. We begin the applications of our model by studying the harvesting of vegetation. The following sequence of figures shows paired phase plane and time series plots of system simulations.

4a. Vegetation Harvesting

We have kept H at 0.08 and model the harvesting of vegetation, by increasing \(h_1\). Based on the algebraic form of the vegetation nullcline, increasing \(h_1\) tends to lower the nullcline. Varying \(h_1\) has no effect on the (vertical) consumer nullcline.

We find that low vegetation harvesting will not prevent the system from crashing at \(h_1= (0.00-0.03)\), Figure 4.1a. In the corresponding time-series plot, Figure 4.1b, an immediate crash occurs for the vegetation and soil nutrient populations and there is a momentary increase in the consumer population before it also crashes. When the harvesting increases to \(h_1=0.04\) the consumer-vegetation populations approach a periodic orbit, Figure 4.2a. There is a larger population fluctuation in the vegetation versus the consumer and soil nutrient over time, Figure 4.2b, shown by the periodic orbit.

The amplitude of the periodic orbit becomes smaller as the harvesting increases, until at \(h_1=0.16\),
Figure 4.3a, an inward spiral to an equilibrium point occurs. The inward stable spiral continues through Figure 4.4a, \( h_1=0.25 \). As \( h_1 \) has increased it has the effect of lowering the coexistence equilibrium for the consumer coordinate. After initial fluctuations in the corresponding time-series plots a stable population is reached for the vegetation-consumer-soil ecosystem as shown in Figures 4.3b and 4.4b.

In Figure 4.5a, \( h_1=0.26 \), and Figure 4.6a, \( h_1=0.39 \), the populations of consumer and vegetation both crash to zero. This illustrates that the increased harvesting of vegetation can cause a sustainable system to become suddenly unsustainable, Figures 4.5b and 4.6b. As the harvesting of vegetation continues the equilibrium values of the vegetation meet at a single point on the vegetation axis for \( h_1 > 0.25 \). For even larger values of \( h_1 \) the vegetation nullcline and the vegetation axis do not intersect, Figure 4.6a, where \( h_1=0.39 \). Since nullclines no longer intersect when \( C > 0 \), it is not possible for the populations to coexist as we witnessed in the preceding Figures 4.1a – 4.5a. Thus, under extreme harvesting the vegetation tends to zero population, as does the consumer. All foregoing phase plane graphs have a vertical consumer nutrient coordinate and a horizontal vegetation nutrient coordinate.
Figure 4.2a: \( h_1 = 0.04; \ (V_0, C_0) = (0.70, 0.05) \)

Figure 4.2b

Figure 4.3a: \( h_1 = 0.16; \ (V_0, C_0) = (0.70, 0.05) \)

Figure 4.3b

Figure 4.4a: \( h_1 = 0.25; \ (V_0, C_0) = (0.70, 0.05) \)

Figure 4.4b
4b. Consumer Harvesting

In the next sequence of figures we are modeling the harvesting of the consumer by increasing $h_2$. The vertical nullcline of the consumer will move in the positive direction on the vegetation axis as the consumer experiences an increase in harvesting. The vegetation nullcline is not affected by $h_2$. An increased vegetation equilibrium value will occur since the rate of growth of the consumer is zero but the consumer population is declining.

Figure 4.7a illustrates the parametric default values where $H=0.8$ and $h_2=0$. As we initiate consumer harvest, $h_2=0.01$, the consumer-vegetation populations lock onto a large periodic orbit around
the intersection of the nullclines. In the time-series plot of Figure 4.8b the vegetation plot has a greater fluctuation in population over time versus the consumer and soil nutrient and is similar to the plot of Figure 4.2b.

When \( h_2 = 0.04 \), Figure 4.9a, there is a tight inward spiral to the stable equilibrium point where the two nullclines intersect. The periodic orbit has been lost due to a reverse Hopf bifurcation at the coexistence equilibrium. The consumer nullcline has moved to the right on the vegetation axis. Our time-series plot shows a transient fluctuation in the populations of the three entities over time, approaching stable equilibrium values as \( t \to +\infty \), Figure 4.9b. Continuing the harvesting of the consumer we notice the vegetation-consumer population curve goes toward the vegetation nullcline and follows it until reaching the equilibrium point where the two nullclines intersect, Figures 4.10a and 4.11a. The soil nutrient equilibrium point is a lesser value than the consumer equilibrium point, Figure 4.10b.

Each of the time-series plots, Figure 4.10b, \( h_2 = 0.08 \), Figure 4.11b, \( h_2 = 0.13 \), and Figure 4.12b, \( h_2 = 0.14 \) shows an immediate movement of the vegetation-consumer population trajectory toward an equilibrium state of the three entities in the ecosystem. In Figure 4.12a, the consumer nullcline has moved to the right on the vegetation axis so the nullclines no longer can intersect. Recall that the vegetation nullcline is not affected by quantities of the consumer harvesting coefficient, \( h_2 \). The equilibrium point ends up becoming the larger valued \( V^* \), which is the carrying capacity of the consumer-free system.
Figure 4.7a: $H=0.8$; $h_2=0.0$; $(V_0,C_0)=(0.7,0.05)$

Figure 4.7b:

Figure 4.8a: $h_2=0.01$; $(V_0,C_0)=(0.70,0.05)$

Figure 4.8b:

Figure 4.9a: $h_2=0.04$; $(V_0,C_0)=(0.70,0.05)$

Figure 1.9b
Figure 4.10a: \( h_2 = 0.08; (V_0, C_0) = (0.70, 0.05) \)

Figure 4.10b: \( C \) (red), \( V \) (green), 0.004S (blue)

Figure 4.11a: \( h_2 = 0.13; (V_0, C_0) = (0.70, 0.05) \)

Figure 4.11b: \( C \) (red), \( V \) (green), 0.004S (blue)

Figure 4.12a: \( h_2 = 0.14; (V_0, C_0) = (0.70, 0.05) \)

Figure 4.12b: \( C \) (red), \( V \) (green), 0.004S (blue)
4c. Soil Fertilization

The next sequence of figures will help to reveal the impact of adding nutrients to the soil, as modeled by increasing $F > 0$. At the default parameter values with $H=0.08$ and $F=0$, the vegetation-consumer population trajectory crashed to zero. By adding fertilization to the soil, $F=0.01$, the solution locks onto a periodic orbit around the intersection of the nullclines, Figure 4.13a. Notice that the vegetation nullcline no longer intersects the positive vegetation axis twice. So fertilization appears to have removed the Allee effect from the system. In the corresponding time-series plot the soil is now fluctuating in population more than the consumer and somewhat less than the vegetation. Figure 4.14a, the soil nutrient addition is 100 times greater than in Figure 4.13a, $F=1.00$. A periodic orbit is found similar to the previous Figure 4.13a but the time series plot (the extended vertical graph) shows the change in quantity of soil nutrient added did not significantly change the equilibrium of the ecosystem. Other experimentation with the simulator for larger $F$ showed little significant change by additional fertilization.

![Figure 4.13a: H=0.08; F=0.01; (V₀,C₀)=(0.70,0.05)](image1)

![Figure 4.13b](image2)
5. Implications of the Model and Topics for Further Study

Although we have considered three harvesting scenarios in the previous section, the methods of this paper allow various “hybrid” cases as well. For example, in the remaining four figures we set \( F=0.01 \), but now we will harvest the vegetation and interpret what kind of effect the fertilization will have on the sustainability of ecosystem. At \( h_1=0.03 \) the vegetation-consumer curve crashed, refer to Figure 4.1a. However, in Figure 5.1a we have set \( h_1=0.03 \) and the addition of soil nutrient has forced the ecosystem into a periodic orbit. The time-series plot has lesser fluctuations of the three entities than in Figure 4.13b but the comparisons in fluctuations are similar. Refer to Figure 4.4a to see the graph for \( h_1=0.16 \), a loose inward spiral. In Figure 5.2a, the inward spiral is tighter and the equilibrium point has a higher consumer value. We need to notice the nullcline for the vegetation is larger than in Figure 4.4a, which allows for the higher consumer value at the stable equilibrium point.

By Figure 4.5a, \( h_1=0.26 \), and Figure 4.6a, \( h_1=0.39 \), we see the vegetation and consumer populations crashing to zero. However, after adding soil nutrient, \( F=0.01 \), Figures 5.1a - 5.4a show that the stable equilibrium points are values other than \((0,0)\). In Figure 5.3a the equilibrium is at the
intersection of the nullclines and in Figure 5.4a the equilibrium is the higher consumer-free equilibrium value, $V^*$, since the nullclines do not intersect at $C > 0$. The time-series plots, Figures 5.3b and 5.4b depict an initial fluctuation in the populations of the three entities, followed by an approach to positive equilibrium points. Thus, the addition of fertilization appears to eliminate the population crash observed in 4.1a, where $F=0$.

**Figure 5.1a:** $h_1=0.03; F=0.01; (V_0,C_0)=(0.70,0.05)$

**Figure 5.1b**

**Figure 5.2a:** $h_1=0.16; F=0.01; (V_0,C_0)=(0.70,0.05)$

**Figure 5.2b**
The mathematical model we have presented furthers our understanding of the dynamics involved in the soil-vegetation-consumer ecosystem. We found that the differential equations we used were sensitive to the parametric values incorporated in our Mathematica simulations. Sudden changes occurred in the approached equilibrium of the ecosystem when we changed harvesting and fertilization values. The time-series plots helped us to understand the fertility of the soil, even though our graphs
were based on a two-dimensional system. Our system was observed to be sensitive to the adding of nutrients. We concluded that fertilization “dampened” the effects of harvesting vegetation and reduced the likelihood of the vegetation tending to zero population. An Allee effect was quite noticeable in some of our simulations, with the model supporting simultaneous stable “no life” state \( \text{V}=\text{C}=0 \) and stable coexistence solutions. The Allee effect was eliminated when fertilization was introduced due to the increased dimensions of the vegetation nullcline.

Our goal has been to derive and parameterize a basic model. Finding the various stability properties for the equilibrium points would be a logical next step for this mathematical model. This is accomplished by computing the eigenvalues for the equilibrium points.

Finally, it should also be observed that our work has involved simulations of the two-dimensional quasi-equilibrium system where it is assumed \( \dot{S} = 0 \). It would be interesting to see how these results compare to simulations of the full three-dimensional system.
6. Appendix: Mathematica Codes

Manipulate[soln1 = NDSolve[{V'[t] == ((d*((F + a*L*V[t])/((α/(1 + β*V[t])))/(b + ((F + a*L*V[t])/((α/(1 + β*V[t]))))/((1 - V[t])^2)) * (Pmax*C[t])/(Pmax*H + V[t]) - h1)*V[t], C'[t] == C[t]*((γ*Pmax*V[t])/(Pmax*H + V[t]) - m - h2), V[0] == V1, C[0] == C1}, {V[t], C[t]}, {t, 0, tmax}, MaxSteps -> 1000000, MaxStepSize -> 0.02];

Show[ContourPlot[0 == ((d*((F + a*L*V))/(α/(1 + β*V)))/(b + ((F + a*L*V)/(α/(1 + β*V)))))*(1 - V/k) - (Pmax*C)/(Pmax*H + V) - h1, 0 == ((γ*Pmax*V)/(Pmax*H + V) - m - h2), V[0] == V1, C[0] == C1}, {V, Vmin, Vmax}, {C, Cmin, Cmax}, PlotPoints -> 40, ContourStyle -> {Darker[Blue], Black, Red}], ParametricPlot[Evaluate[{V[t], C[t]}/.soln1], {t, 0, tmax}, Axes -> True, PlotStyle -> Black, AxesLabel -> "V", "C"], PlotRange -> {Vmin, Vmax}, {Cmin, Cmax}], PlotPoints -> 50, Graphics[{PointSize[.01], Red, Point[{V1, C1}]}],

Style["Model Parameters", Bold], {{k, .95}, 0, 5, .001}, {{b, .04}, 0, .01}, {{d, 1}, 0, .01}, {{F, 0}, 0, .01}, {{a, .093}, 0, .1}, {{L, .4}, 0, 1, .01}, {{α, .01}, 0, 1.2, .01}, {{β, 100}, .01, 200, .01}, {{h1, 0}, 0, 1, .01}, {{h2, 0}, 0, 1, .01}, {{Pmax, 1}, 0, 10}, {{H, 0.6}, 0, 1, .01}, {{γ, .9}, 0, 1, .01}, {{m, .68}, 0, 1, .01}, Delimiter, Style["Initial Conditions / Simulation Length", Bold], {{V1, .7}, 0, Vmax, .01}, {{C1, .05}, 0, Cmax, .01}, {{tmax, 300}, 1, 500, 1}, Delimiter, Style["Window Size:", Bold], {{Vmax, .85}, 1, 1, .01}, {{Cmax, .2}, 0, 5, .001}, {{Vmin, .0001}, 0, 30, .1}, {{Cmin, .01}, 1, 1, .01}, ControlPlacement -> Right]
Manipulate[soln2 = NDSolve[
{V'[t] == ((d*(F + a*L*V[t])/(α/(1 + β*V[t]))))/((b + ((F + a*L*V[t])/(α/(1 + β*V[t])))))*V[t]/k) - (Pmax*C[t])/((Pmax*H + V[t] - L - h1) - V[t]),
C'[t] == C[t] * ((γ*Pmax*V[t])/(Pmax*H + V[t] - m - h2)),
V[0] == V1, C[0] == C1}, {V[t], C[t]}, {t, 0, tmax}, MaxSteps -> 1000000, MaxStepSize -> 0.02];
Plot[Evaluate[{V[t], C[t], .004*((F + a*L*V[t])/(α/(1 + β*V[t])))} /. soln2], {t, tmin, tmax}, Axes -> True, PlotStyle -> {Dark[Green], Darker[Red], Darker[Blue]}, AxesLabel -> {"t", "C (red), V (green), .004*S (blue)"}, PlotRange -> {{tmin, tmax}, {ymin, ymax}}, PlotPoints -> 50];
Style["Model Parameters", Bold], {{k, .95}, 0.5, 0.001}, {{b, .04}, 0.3, 0.01}, {{d, 1}, 0.3, 0.01}, {{F, 0}, 0.1, 0.01}, {{a, .093}, 0.1, 0.01}, {{L, 4}, 0.1, 0.01}, {{α, 0.01}, 0.1, 2.0, 0.01}, {{β, 100}, 0.1, 200, 0.01}, {{h1, 0}, 0.1, 0.01}, {{h2, 0}, 0.1, 0.01}, {{Pmax, 1}, 0, 10}, {{H, .06}, 0.1, 0.01}, {{γ, 9}, 0.1, 0.01}, {{m, .68}, 0.1, 0.01}, Delimiter, Style["Initial Conditions / Simulation Length", Bold], {{V1, 3}, 0.3, 0.01}, {{C1, 0.5}, 0.1, 0.01}, {{tmin, 0}, 0.5, 0.01}, {{tmax, 150}, 1.5, 0.01}, Delimiter, Style["Window Size:", Bold], {{ymax, .6}, 0.1, 2.0, 0.01}, {{ymin, -0.1}, -1.1, 0.01}, ControlPlacement -> Right]
7. References


