Graph labelings and multipartite decompositions

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A decomposition of a graph $G$ is a set $\mathcal{H}$ of subgraphs of $G$ whose edge sets partition the edge set of $G$. If $\mathcal{H} = \{H\}$, we say that $G$ has an $H$-decomposition. If all sets in $\mathcal{H}$ are bipartite ($r$-partite) we say that $\mathcal{H}$ is a bipartite ($r$-partite) decomposition of $G$. The minimum biclique cover $[3]$ is a well-studied graph decomposition problem that asks for a minimum set of bicliques required to decompose a given graph. More recently the minimum bipartite graph cover problem has been introduced $[2]$. Thus the question, how many bipartite subgraphs are required to decompose a given graph, is very relevant.

Graph labelings have been frequently used to study decompositions. For instance the classical result by Rosa $[1, 4]$ states that if a graph $G$ with $q$ edges allows a rosy labeling, then it decomposes $K_{2q+1}$, and if $G$ allows an $\alpha$-labeling, then it decomposes $K_{2pq+1}$ for every $p > 0$. In this paper we use a different type of vertex labeling to study bipartite decompositions and prove that

**Theorem 1** Let $k$ be any positive integer. If a graph $G$ can be decomposed into $k$ bipartite graphs then it is $2^k$-colorable. Conversely, if $G$ is $2^k$-colorable then it can be decomposed into $k$ or less bipartite graphs.

We then investigate $r$-partite decompositions for positive integer $r > 2$ and generalize Theorem 1.

**Theorem 2** Let $k$ and $r$ be any positive integers. If a graph $G$ can be decomposed into $k$ $r$-partite graphs then it is $r^k$-colorable. Conversely, if $G$ is $r^k$-colorable then it can be decomposed into $k$ or less $r$-partite graphs.

Several important consequences of Theorem 1 and Theorem 2 are also discussed.

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References


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