PATTERNS IN CONTINUED FRACTION EXPANSIONS

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<u>Chapter 1</u> - Introduction

It is well known that any real number has a unique (or almost unique) decimal expansion. Since we do not typically write an infinite string of zeros down, these expansions can be either finite or infinite. For instance in base 10, 31/25 has decimal expansion 1.24, 1/3 has decimal expansion $0.3333\ldots = 0.\overline{3}$, and π has decimal expansion 3.14159…… However, in base 3 the decimal expansions of 31/25, 1/3, and π are 1.020110221221, 0.1, and 10.0102110122… respectively. Notice that not only do the decimal expansions change with different bases, but also whether the expansion is finite or infinite. Real numbers have another interesting expansion called a *continued fraction expansion.* In a sense, the continued fraction expansion of a real number is base independent. Since these expansions are given by listing nonnegative integers, when we consider expansions in different bases the only thing that changes is how we represent those integers. Whether or not the expansion is finite or infinite does not change, even if we do change the base. For example, in base 10, 31/25 has continued fraction expansion [1,4,6], the expansion of 1/3 is [0,3], and the expansion for π is[3,7,15,1,…]. In base 3, the expansions of 31/25, 1/3, and π are [1,11,20], [0,10], and [10,21,120,1,…]. These expansions are unique, with one exception. Continued fraction expansions are much different than decimal expansions and the expansion alone can provide us with a considerable amount of information. In this regard, representing numbers as continued fractions is more beneficial than using a decimal system. However, it does have drawbacks as even operations such as addition are extremely difficult to perform on two continued fraction expansions [3, p.p.19-20].To understand what this expansion is, we must first define a *continued fraction.*

<u>Definition 1:</u> An expression of the form

(1)
$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}}$$

where a_i, b_i are real or complex numbers is called a *continued fraction.* An expression of the form

(2)
$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where $b_i = 1$ for all i, a_0 is an integer, and $a_0, a_1, a_2, ...$ are each positive integers is called a *simple continued fraction*. Due to the cumbersome nature of the notation above, it is more common to express (2) as $a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}\cdots$ or simply as $[a_0, a_1, a_2, a_3, ...]$. We will mostly use the latter of the expressions, and we sometimes refer it to as the continued fraction expansion of a number. The terms $a_0, a_1, a_2, ...$ are called *partial quotients*. If there are a finite number of partial quotients, we call it a *finite simple continued fraction*, otherwise it is *infinite*. In this paper when we refer to *continued fractions*, we really are referring to *simple continued fractions*, the only continued fraction we consider.

As an example of a continued fraction, let's calculate the continued fraction expansion of a rational number.

Example 1. To find the continued fraction expansion of $\frac{43}{19}$ we can proceed as follows:

$$\frac{43}{19} = 2 + \frac{5}{19} = 2 + \frac{1}{\frac{19}{5}} = 2 + \frac{1}{3 + \frac{4}{5}} = 2 + \frac{1}{3 + \frac{1}{\frac{5}{4}}} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1}}}}$$

We can see from this that both the last two expressions match expression (2). Hence this example shows us that $\frac{43}{19}$ has two continued fraction expansions, $[2,3,1,4]$ and $[2,3,1,3,1]$. This leads us to our first theorem.

Theorem 1 [6, p.14]. Any finite continued fraction represents a rational number, and any rational number can be represented as a finite continued fraction. Furthermore, this continued fraction is unique, apart from the identity $[a_0, a_1, a_2, ..., a_n] = [a_0, a_1, a_2, ..., a_n - 1, 1]$.

Although in example 1 we showed a method of calculating the continued fraction expansion of a number, it would be nice to have a systematic approach to finding the expansion of any real number, not just rational ones. *The continued fraction algorithm* gives us just that.

The Continued Fraction Algorithm

Suppose we wish to find the continued fraction expansion of $x \in \Re$. We proceed as follows. Let $x_0 = x$ and set $a_0 = \lfloor x_0 \rfloor$. We then define $x_1 = \dfrac{1}{x_0 - \lfloor x_0 \rfloor}$ and set $a_1 = \lfloor x_1 \rfloor$. We proceed in this manner;

$x_2 = \dfrac{1}{x_1 - \lfloor x_1 \rfloor} \Rightarrow a_2 = \lfloor x_2 \rfloor, \ldots,\ x_k = \dfrac{1}{x_{k-1} - \lfloor x_{k-1} \rfloor} \Rightarrow a_k = \lfloor x_k \rfloor, \ldots$ We either continue indefinitely, or we stop if we find a value $x_i \in \mathbb{N}$ [5, p.p.229-230].

To illustrate this algorithm, consider the following example.

Example 2 We shall calculate the continued fraction expansion of $\dfrac{414}{283} \approx 1.4629$.

Let $x_0 = \dfrac{414}{283}$, so $a_0 = 1$.Then

$$x_1 = \frac{1}{\frac{414}{283} - 1} = \frac{283}{131} \approx 2.1603 \Rightarrow a_1 = 2,$$

$$x_2 = \frac{1}{\frac{283}{131} - 2} = \frac{131}{21} \approx 6.2381 \Rightarrow a_2 = 6,$$

$$x_3 = \frac{1}{\frac{131}{21} - 6} = \frac{21}{5} = 4.2000 \Rightarrow a_3 = 4,$$

$$x_4 = \frac{1}{\frac{21}{5} - 4} = 5 \Rightarrow a_4 = 5.$$

Since $x_4 = 5$, we are done. Thus we conclude that $\dfrac{414}{283} = [1, 2, 6, 4, 5]$.

As mentioned above, the continued fraction algorithm can be applied to irrational numbers as well. As a consequence of Theorem 1, the algorithm, when applied to an irrational number, will continue indefinitely. Some irrational numbers, square roots for example, have continued fraction expansions that exhibit nice periodic behavior. Other numbers such as e have evident patterns that occur in their expansions, and yet others such as π have expansions that do no not appear to follow any patterns. Below are some examples along with their continued fraction expansions.

$$\begin{aligned}
&\sqrt{3} = [1,1,2,1,2,1,2,...] = [1,\overline{1,2}]\\
&\sqrt{7} = [2,1,1,1,4,1,1,1,4,1,1,1,4,...] = [2,\overline{1,1,1,4}]\\
&e = [2,1,2,1,1,4,1,1,6,1,1,8,1...]\\
&\pi = [3,7,15,1,292,1,1,1,2,1,...]
\end{aligned}$$

It is difficult to prove the above expansions of π or e, however the next example illustrates that one can find the expansion of $\sqrt{3}$ with ease.

Example 3 We follow the continued fraction algorithm. Let $x_0 = \sqrt{3}$. Since $1 < \sqrt{3} < 2$, $a_0 = 1$.

Now,

$$x_1 = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} \Rightarrow a_1 = \left\lfloor \frac{\sqrt{3}+1}{2} \right\rfloor = 1,$$

$$x_2 = \frac{1}{\frac{\sqrt{3}+1}{2}-1} = \frac{2}{\sqrt{3}-1} = \frac{2\sqrt{3}+2}{2} = \sqrt{3}+1 \Rightarrow a_2 = 2,$$

$$x_3 = \frac{1}{\sqrt{3}+1-2} = \frac{1}{\sqrt{3}-1} = x_1.$$

Since $x_3 = x_1$ this clearly forces $x_4 = x_2, x_5 = x_1, ..., x_{2k} = x_2, x_{2k+1} = x_1, ...$ and so the corresponding partial quotients alternate between 1 and 2 indefinitely. Therefore, $\sqrt{3} = [1,\overline{1,2}]$.

Uses of Continued Fractions

Continued fractions constitute a major branch of number theory because they have many applications within the field. First of all, they provide us with a method to find the best rational approximations of a real number in the sense that no other rational with a smaller denominator is a better approximation [3, p.p.26-28]. Continued fractions allow one to find solutions of linear Diophantine equations with ease. See [6, p.p.31-46]. Also the continued fraction expansion of $\sqrt{n}$ can be used to find solutions to Pell's equation, $x^2 - ny^2 = 1$. For more information on Pell's Equation and continued fractions, refer to [2]. Furthermore, continued fractions can be put to use in the factorization of large integers [5, p.246]. We can also make use of continued fractions to help prove that any prime p of the form $4k+1$ can be expressed uniquely as the sum of two squares [6, p.p.132-133].

Motivation of our problem

This paper was inspired by the following question. Suppose we start with some number x which has known expansion $[a_0, a_1, a_2, ...]$ and we add to it a decreasing sequence of positive values r_n. Then as the value of $x + r_n$ approaches x what happens to the corresponding continued fraction expansion? It turns out that some interesting patterns become evident. To illustrate this consider the following:

Example 4 Let $x = 1 + 2^{-2} + 2^{-9}$ which has expansion $[1,3,1,31,4]$. Now the following table gives the expansions of the numbers $1 + 2^{-2} + 2^{-9} + 2^{-i}$ for $10 \le i \le 22$.

i	Continued Fraction Expansion
10	[1, 3, 1, 20, 1, 1, 2, 2]
11	[1, 3, 1, 24, 1, 5, 1, 2]
12	[1, 3, 1, 27, 1, 2, 3, 1, 2]
13	[1, 3, 1, 29, 2, 1, 2, 1, 1, 3]
14	[1, 3, 1, 30, 3, 1, 1, 3, 5]
15	[1, 3, 1, 30, 1, 3, 7, 1, 7]
16	[1, 3, 1, 31, 516]
17	[1, 3, 1, 31, 7, 1, 31, 4]

18	[1, 3, 1, 31, 5, 3, 31, 1, 3]
19	[1, 3, 1, 31, 4, 1, 1, 3, 31, 1, 3]
20	[1, 3, 1, 31, 4, 3, 1, 3, 31, 1, 3]
21	[1, 3, 1, 31, 4, 7, 1, 3, 31, 1, 3]
22	[1, 3, 1, 31, 4, 15, 1, 3, 31, 1, 3]

Table 1

Looking at the table we see that the $[1,3,1,...]$ pattern appears in each expansion and when $i \geq 16$ each pattern starts with $[1,3,1,31,...]$, the first 4 partial quotients of x. For $i \geq 18$, the entire expansion of x appears in the beginning. However this isn't the only interesting thing to make note of. For $i \geq 19$ we also see that the only partial quotient that changes is the one immediately following the 4 while the rest of the partial quotients are $[1,3,31,1,3]$. If we look at this portion in reverse order, we see that $[3,1,31,3,1] = [3,1,31,4]$ which exactly matches all but the first term of the expansion of x. In this paper, we will look into other interesting patterns that arise in continued fraction expansions and explain when precisely this pattern occurs and why it does. A particularly nice result that came about from this investigation can be found in chapter 4. It gives explicitly the continued fraction expansion for a rational number of the form $\frac{p}{q} + \frac{(-1)^n}{(k+1)q^2}$ for k a nonnegative integer, given the expansion of $\frac{p}{q}$.

We pause here to make a quick note regarding the Theorems discussed in this paper. The first nine theorems are commonly found in any textbook on continued fractions. Theorems 10 and those that follow are introduced in this paper. That Theorem 17 exists, however, is hinted at in [5, p.238].

Chapter 2 - Properties and Important Relations

One essential tool in studying the theory of continued fractions is the study of the *convergents* of a continued fraction.

Definition 2: Let $x=[a_0,a_1,a_2,...,a_n,...]$. The reduced fractions given below are called the *convergents* of x and are defined by:

$$\frac{p_0}{q_0}=a_0\ ,\quad \frac{p_1}{q_1}=a_0+\frac{1}{a_1},\quad \frac{p_2}{q_2}=a_0+\frac{1}{a_1+}\frac{1}{a_2},\quad \cdots\quad ,\ \frac{p_n}{q_n}=a_0+\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+\ldots+}\frac{1}{a_n}\ \cdots.$$

Theorem 2 [5, p.233]. Let $p_0,p_1,p_2,...$ denote the numerators of the convergents of some number x while $q_0,q_1,q_2,...$ denotes the denominators. Now define $\begin{cases} p_{-2}=0, & p_{-1}=1 \\ q_{-2}=1, & q_{-1}=0 \end{cases}$ and define x_i as in the continued fraction algorithm. Then the following relations hold.

i. $\begin{cases} p_k=a_k p_{k-1}+p_{k-2} \\ q_k=a_k q_{k-1}+q_{k-2} \end{cases}$ for $k\ge 0$

ii. $p_{k-1}q_k-p_kq_{k-1}=(-1)^k$ for $k\ge -1$

iii. $x=\dfrac{p_{k-1}x_k+p_{k-2}}{q_{k-1}x_k+q_{k-2}}$ for $k\ge 0$

iv. $x_k=-\dfrac{p_{k-2}-q_{k-2}x}{p_{k-1}-q_{k-1}x}$

Proof

We prove (ii.) and (iii.). Property (iv.) follows directly from property (iii).

Proof of (ii): This result follows by making use of relation (i) and induction. To establish a basis for induction, we use the given initial values to show the relation holds for $k=-1, k=0$, and $k=1$.

$$p_{-2}q_{-1}-p_{-1}q_{-2}=0\cdot 0-1\cdot 1=(-1)^{-1}$$

$$p_{-1}q_0 - p_0 q_{-1} = 1 \cdot 1 - a_0 \cdot 0 = (-1)^0$$

$$p_0 q_1 - p_1 q_0 = a_0 a_1 - (a_0 a_1 + 1) \cdot 1 = (-1)^1$$

Now suppose it holds for some integer $m \geq 3$. Then,

$$\begin{aligned}
&p_m q_{m+1} - p_{m+1} q_m \\
&\quad = p_m (a_{m+1} q_m + q_{m-1}) - (a_{m+1} p_m + p_{m-1}) q_m \\
&\quad = p_m q_{m-1} - p_{m-1} q_m \\
&\quad = -1(p_{m-1} q_m - p_m q_{m-1}) \\
&\quad = -1(-1)^m && \text{by our induction assumption} \\
&\quad = (-1)^{m+1}
\end{aligned}$$

So it holds for m+1 as well.

<u>*Proof of (iii):*</u> Again we proceed by induction. Recall from the continued fraction algorithm that

$$x_0 = x \qquad \text{and} \qquad x_k = \frac{1}{x_{k-1} - \lfloor x_{k-1} \rfloor} = \frac{1}{x_{k-1} - a_{k-1}}$$

For $k = 0$,

$$\frac{p_{-1} x_0 + p_{-2}}{q_{-1} x_0 + q_{-2}} = \frac{1 \cdot x + 0}{0 \cdot x + 1} = x$$

For $k = 1$,

$$\frac{p_0 x_1 + p_{-1}}{q_0 x_1 + q_{-1}} = \frac{a_0 \left(\frac{1}{x - a_0} \right) + 0}{1 \cdot \left(\frac{1}{x - a_0} \right) + 0} \bullet \frac{x - a_0}{x - a_0} = \frac{a_0 + x - a_0}{1} = x.$$

So the result holds for $k = 0$ and $k = 1$. Assume it holds for some number $n \geq 2$, then we have

$$\frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}} = \frac{p_n\left(\frac{1}{x_n - a_n}\right) + p_{n-1}}{q_n\left(\frac{1}{x_n - a_n}\right) + q_{n-1}} \bullet \frac{x_n - a_n}{x_n - a_n}$$

$$= \frac{p_n + p_{n-1}(x_n - a_n)}{q_n + q_{n-1}(x_n - a_n)} = \frac{a_n p_{n-1} + p_{n-2} + p_{n-1}(x_n - a_n)}{a_n q_{n-1} + q_{n-2} + q_{n-1}(x_n - a_n)}$$

$$= \frac{p_{n-1}x_n + p_{n-2}}{q_{n-1}x_n + q_{n-2}} = x \qquad \text{by the induction assumption.}$$

So the result holds for n+1 as well.

$\square$

Applying property (i) of Theorem 2 can give us an efficient way of calculating the convergents of a continued fraction if we know the partial quotients. Example 5 demonstrates this.

<u>Example 5</u> Consider $\frac{1380}{1051} = [1,3,5,7,9]$. We can calculate the convergents by using the following table:

i	-2	-1	0	1	2	3	4
a_i			1	3	5	7	9
p_i	0	1	1	4	21	151	1380
q_i	1	0	1	3	16	115	1051

Notice that if we follow the arrows in the diagram above, to find $p_3 = 151$ we multiply 7 by 21 and add 4. Similarly, to find $q_4 = 1051$ we multiply 9 by 115 and add 16 to it. We can also use the table above to illustrate properties (ii) and (iii) from Theorem 2. For property (ii), we see that $p_2q_3 - p_3q_2 = 21(115) - 151(16) = -1$. To demonstrate property (iii) for say, x_2 and x_3, first observe that

$x = x_0 = \frac{1380}{1051}$ so $x_1 = \frac{1}{\frac{1380}{1051} - 1} = \frac{1051}{329}$ and $x_2 = \frac{1}{\frac{1051}{329} - 3} = \frac{329}{64}$. Now,

$$\frac{p_1x_2+p_0}{q_1x_2+q_0}=\frac{4\left(\frac{329}{64}\right)+1}{3\left(\frac{329}{64}\right)+1}=\frac{1380}{1051} \qquad \text{and} \qquad \frac{p_2x_3+p_1}{q_2x_3+q_1}=\frac{21\left(\frac{64}{9}\right)+4}{16\left(\frac{64}{9}\right)+3}=\frac{1380}{1051}.$$

The following is an interesting and useful result.

Theorem 3 [6, p.26]. If $\frac{p_n}{q_n}=[a_0,a_1,a_2,...,a_n]$ then $\frac{q_k}{q_{k-1}}=[a_k,...,a_2,a_1]$ for $1\le k\le n$ .Also if $a_0\neq 0$ then $\frac{p_k}{p_{k-1}}=[a_k,...,a_2,a_1,a_0]$. If $a_0=0$ then for $2\le k\le n$ $\frac{p_k}{p_{k-1}}=[a_k,...,a_4,a_3,a_2]$.

Proof

Making use of Theorem 2(i) for $k=1$ we have:

$$\frac{q_1}{q_0}=\frac{a_1q_0+q_{-1}}{q_0}=a_1+\frac{q_{-1}}{q_0}=a_1+\frac{0}{1}=a_1.$$

For $k=2$ we have:

$$\frac{q_2}{q_1}=\frac{a_2q_1+q_0}{q_1}=a_2+\frac{q_0}{q_1}=a_2+\frac{1}{\frac{q_1}{q_0}}=a_2+\frac{1}{a_1}.$$

Now assume the result holds for some integer $m\ge 2$. So,

$$\frac{q_m}{q_{m-1}}=[a_m,a_{m-1},...,a_1]=a_m+\cfrac{1}{a_{m-1}+\cfrac{1}{\ddots+\cfrac{1}{a_1}}}.$$

Now,

$$\frac{q_{m+1}}{q_m}=\frac{a_{m+1}q_m+q_{m-1}}{q_m}$$

$$=a_{m+1}+\frac{q_{m-1}}{q_m}=a_{m+1}+\frac{1}{\frac{q_m}{q_{m-1}}}$$

$$= a_{m+1} + \cfrac{1}{a_m + \cfrac{1}{a_{m-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}}$$ by the induction assumption.

Thus the result holds for $m+1$ so it holds for $1 \le k \le n$ by induction. The proof for the $\frac{p_k}{p_{k-1}} = [a_k, ..., a_2, a_1, a_0]$ case is similar.

□

Theorem 4 [6, p.70]. Every infinite continued fraction $[a_0, a_1, a_2, ..., a_n, ...]$ uniquely represents an irrational number y. Conversely, if y is an irrational number then its continued fraction expansion is infinite.

It is important to note a few things regarding Theorem 4. First, as mentioned earlier, from Theorem 1 we immediately know that an irrational number will have an infinite continued fraction expansion. Next, we need to clarify what is meant when we say the expression $[a_0, a_1, a_2, ..., a_n, ...]$ represents an irrational number. Saying that $y = [a_0, a_1, a_2, ..., a_n, ...]$ means that $\lim_{n\to\infty} \frac{p_n}{q_n} = y$. At this stage, we should provide some justification for this. First, observe that for any nonnegative integer n,

$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-2}}{q_{2n-2}} = \frac{p_{2n}q_{2n-2} - p_{2n-2}q_{2n}}{q_{2n}q_{2n-2}}$$

$$= \frac{(a_{2n}p_{2n-1} + p_{2n-2})q_{2n-2} - p_{2n-2}(a_{2n}q_{2n-1} + q_{2n-2})}{q_{2n}q_{2n-2}}$$ By Theorem 2 (i)

$$= \frac{a_{2n}(p_{2n-1}q_{2n-2} - p_{2n-2}q_{2n-1})}{q_{2n}q_{2n-2}}$$

$$= \frac{a_{2n}(-1)^{2n-2}}{q_{2n}q_{2n-2}}$$ by Theorem 2 (ii)

$$= \frac{a_{2n}}{q_{2n}q_{2n-2}}.$$

In a similar manner, it can be shown that

$$\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{-a_{2n}}{q_{2n+1}q_{2n-1}}.$$

These results tell us that the even convergents form an increasing sequence while the odd convergents form a decreasing sequence. That is,

(3) $$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_{2n}}{q_{2n}} < \ldots \quad \text{and} \quad \ldots < \frac{p_{2n+1}}{q_{2n+1}} < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Applying Theorem 2 (ii) for any nonnegative integer n we also have,

$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{(-1)^{2n-1}}{q_{2n}q_{2n-1}} = \frac{-1}{q_{2n}q_{2n-1}}$$

(4) $$\Rightarrow \frac{p_{2n}}{q_{2n}} < \frac{p_{2n-1}}{q_{2n-1}},$$

so the even convergents are less than the odd convergents. Combining (3) and (4) now tells us that:

(5) $$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_{2n}}{q_{2n}} < \ldots < \frac{p_{2n-1}}{q_{2n-1}} < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Now the sequences $\left\{\frac{p_{2n}}{q_{2n}}\right\}$ and $\left\{\frac{p_{2n+1}}{q_{2n+1}}\right\}$ are both monotonic and bounded, and therefore are convergent. Furthermore, they are subsequences of $\left\{\frac{p_n}{q_n}\right\}$. Finally, observe that

$\lim_{n\to\infty}\left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n+1}}{q_{2n+1}}\right) = \lim_{n\to\infty}\left(\frac{-1}{q_{2n}q_{2n+1}}\right) = 0$ since $q_n \to \infty$ as $n \to \infty$. Hence, the sequence $\left\{\frac{p_n}{q_n}\right\}$ is a Cauchy sequence and therefore converges to some irrational number, say y.

Now since $\left\{\frac{p_n}{q_n}\right\}$ converges to y so do $\left\{\frac{p_{2n}}{q_{2n}}\right\}$ and $\left\{\frac{p_{2n+1}}{q_{2n+1}}\right\}$. This leads us to the following:

Theorem 5 [6, p.63]. The even convergents of the continued fraction expansion of y are all less than y and they form an increasing sequence. The odd convergents of y are all greater than y and they form a decreasing sequence. That is,

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \frac{p_{2n}}{q_{2n}} < \ldots < y < \ldots < \frac{p_{2n-1}}{q_{2n-1}} < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Important Remark: Suppose α is some real number and $\alpha = [a_0, a_1, a_2, ..., a_{k-1}, ...]$. If we let $\alpha = [a_0, a_1, a_2, ..., a_{k-1}, \alpha_k]$ where $\alpha_k = a_k + \frac{1}{a_{k+1} +} \frac{1}{a_{k+2} + \cdots}$ and if we define α_k analogously to x_k from the continued fraction algorithm, then Theorem 2 (iii) still holds. The proof given was independent of whether x_k was rational or irrational. That is, $\alpha = \frac{p_{k-1}\alpha_k + p_{k-2}}{q_{k-1}\alpha_k + q_{k-2}}$. The upshot of this is that if we write any real number α as $[a_0, a_1, a_2, ..., a_{k-1}, \alpha_k]$ then we can still apply all the properties of Theorem 2 just as if α_k were a_k even though α_k can be any real number greater than or equal to 1. We refer to α_k as a *complete quotient* [5, p.231].

Theorem 6 (***Lagrange's Theorem)*** [1, p.144]. Any quadratic irrational number α has a continued fraction expansion which is periodic from some point onward. Conversely, if we start with a continued fraction expansion that is eventually periodic, then it represents a quadratic irrational number.

We will just provide a sketch of the proof. For a more detailed proof see [1, p.p. 144-145].

Proof Sketch Suppose x is a real number with a continued fraction expansion that is eventually periodic with a period length of l. That is, $x = [a_0, a_1, ..., \overline{a_k, a_{k+1}, ..., a_{k+l-1}}]$. If $x_k = [\overline{a_k, a_{k+1}, ..., a_{k+l-1}}]$ then $x_k = [\overline{a_{k+l}, a_{k+l+1}, ..., a_{k+2l-1}}]$ as well. Now by Theorem 2(iii) we have,

$$x = \frac{p_{k-1}x_k + p_{k-2}}{q_{k-1}x_k + q_{k-2}} = \frac{p_{k+l-1}x_k + p_{k+l-2}}{q_{k+l-1}x_k + q_{k+l-2}}$$

It is clear from above that x_k is irrational and that x_k satisfies a quadratic equation with integral coefficients. By substituting $x_k = -\frac{p_{k-2} - q_{k-2}x}{p_{k-1} - q_{k-1}x}$ from Theorem 2(iv), it becomes evident that x is a quadratic irrational as well.

To prove the converse, suppose that x is some quadratic irrational. Hence, x satisfies the quadratic equation

$$Ax^2 + Bx + C = 0,$$

for some integers *A*, *B*, and *C*. Once again by Theorem 2 parts (iii) and (iv), $x = \frac{p_{k-1}x_k + p_{k-2}}{q_{k-1}x_k + q_{k-2}}$ for $k \geq 0$ and $x_k = -\frac{p_{k-2} - q_{k-2}x}{p_{k-1} - q_{k-1}x}$. Thus x_k is a quadratic irrational as well and so it satisfies the equation

$$A_k x_k^2 + B_k x_k + C_k = 0,$$

where the integers A_k, B_k ,and C_k are each defined in terms of the integers *A*, *B*, and *C* respectively. From here, it can be shown that A_k, B_k, and C_k are all bounded by some constant, say *m*, independent of *k*. Therefore, there can only be a finite number of different triples (A_k, B_k, C_k), and hence we can find 3 distinct indices, say n_1, n_2 ,and, n_3 such that $(A_{n_1}, B_{n_1}, C_{n_1}) = (A_{n_2}, B_{n_2}, C_{n_2}) = (A_{n_3}, B_{n_3}, C_{n_3})$. So x_{n_1}, x_{n_2}, and x_{n_3} are three roots of the quadratic equation corresponding to this triple, which means that two of them must be the same. Since a_k is determined directly from x_k, if $x_{n_1} = x_{n_2}$ say, then its expansion must be periodic from that point on.

<u>Chapter 3</u> – Approximation

It is often very practical to approximate irrational numbers with rational numbers. It is clear that given any irrational number, we can approximate it with a rational number to any desired accuracy. The more accurate an approximation we desire, the larger the denominator of the rational must be. The convergents of a continued fraction provide us with a method to find rational numbers that approximate irrational numbers while having as small a denominator as possible. In fact, no other rational numbers with smaller denominators can approximate irrational numbers better than its convergents. For example, consider $\pi = 3.141592653...$ which has expansion $[3,7,15,1,292,1,1,1,2,1,...]$. We can easily find that $\frac{3141592}{1000000} = \frac{392699}{125000} = 3.141592$ matches the first 7 digits of the decimal expansion of π. However, if we look at the convergents of π we see that one of them is $[3,7,15,1] = \frac{355}{113}$ which has decimal expansion 3.1415929… which is actually not only a better approximation for π but also has a denominator that is a great deal smaller. No rational number with a denominator smaller than 113 can provide a better approximation to π.

In this paper, we investigate quantities of the form $\frac{p}{q} + r$. When r happens to be irrational, we can use the approximation properties of the convergents of r to help assist us in studying the continued fraction expansion of $\frac{p}{q} + r$. Some classic theorems on continued fractions and approximation are given below. If you wish to learn more about approximation using continued fractions, see [4] or [1, p.p. 154-176].

<u>Theorem 7</u> [6, p.72]. Let y be an irrational number and $\frac{p_k}{q_k}, \frac{p_{k+1}}{q_{k+1}}$ be successive convergents of y. Then

$$\left| y - \frac{p_{k+1}}{q_{k+1}} \right| < \left| y - \frac{p_k}{q_k} \right| .$$

Furthermore, at least one, say $\frac{p}{q}$, satisfies the inequality:

$$\left|y-\frac{p}{q}\right|<\frac{1}{2q^2}.$$

<u>Corollary</u> If x is irrational, then there exists an infinite number of rationals $\frac{p}{q}$ such that

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^2}.$$

The next theorem is extremely interesting and somewhat surprising, as it tells us that if a rational number approximates an irrational number well enough, then it must be one of its convergents.

<u>Theorem 8</u> [5, p.p.237-238]. For any real number β , if

$$\left|\beta-\frac{p}{q}\right|<\frac{1}{2q^2}$$

Then $\frac{p}{q}$ is necessarily one of the convergents of the continued fraction expansions of β .

The following theorem gives upper and lower bounds on the distance between an irrational number and any of its convergents.

<u>Theorem 9</u> [5, p.237]. If α is irrational then for any $k \geq 0$,

$$\frac{1}{q_n(q_n+q_{n+1})}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}}.$$

<u>Proof</u> Let $\alpha=[a_0,a_1,...,a_n,\alpha_{n+1}]$ where $\alpha_{n+1}=[a_{n+1},a_{n+2},...]$. Then by Theorem 2 parts (ii) and (iii) we have,

$$\alpha=\frac{\alpha_{n+1}p_n+p_{n-1}}{\alpha_{n+1}q_n+q_{n-1}} \text{ and so } \left|\alpha-\frac{p_n}{q_n}\right|=\left|\frac{(-1)^n}{q_n(\alpha_{n+1}q_n+q_{n-1})}\right|=\frac{1}{q_n(\alpha_{n+1}q_n+q_{n-1})}.$$

Now observe that

$$q_{n+1}=a_{n+1}q_n+q_{n-1}<\alpha_{n+1}q_n+q_{n-1}$$

$$\Rightarrow \frac{1}{q_n(\alpha_{n+1}q_n + q_{n+1})} < \frac{1}{q_n q_{n+1}} \text{ ,}$$

and also that

$$\alpha_{n+1}q_n + q_{n-1} < \left(a_{n+1} + 1\right)q_n + q_{n-1} = \left(a_{n+1}q_n + q_{n-1}\right) + q_n = q_{n+1} + q_n$$

$$\Rightarrow \frac{1}{q_n(\alpha_{n+1}q_n + q_{n+1})} > \frac{1}{q_n(q_n + q_{n+1})} \text{ .}$$

This completes the proof.

Corollary [5, p.237] If α is irrational then,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \text{ .}$$

Chapter 4 - Patterns in Continued Fraction Expansions

As previously mentioned, the goal of this paper is to investigate how the continued fraction expansion of $\frac{p}{q}+r$ changes as the value of r changes. We begin this chapter by investigating the continued fraction expansions of various infinite series. Let's first consider the simple geometric series $1+\frac{1}{q}+\frac{1}{q^2}+\frac{1}{q^3}+\ldots=\sum_{n=0}^{\infty} q^{-n}$ where q is an integer greater than 1. The ratio of terms in this series is $\frac{1}{q}$ so we know this series converges to $\frac{1}{1-\frac{1}{q}}=\frac{q}{q-1}$. In the next example we show how the continued fraction expansions of the partial sums can be used to derive this.

Example 6 The first two partial sums of $\sum_{n=0}^{\infty} q^{-n}$ clearly have expansions $[1]$ and $[1,q]$. Now let's find the expansion of $1+\frac{1}{q}+\frac{1}{q^2}$.

$$1+\frac{1}{q}+\frac{1}{q^2}=1+\frac{q+1}{q^2}=1+\frac{1}{\frac{q^2}{q+1}}=1+\frac{1}{\frac{q^2-1+1}{q+1}}=1+\frac{1}{\frac{(q+1)(q-1)+1}{q+1}}=1+\frac{1}{q-1+\frac{1}{q+1}},$$

thus $1+\frac{1}{q}+\frac{1}{q^2}=[1,q-1,q+1]$. In the same manner, we shall now find the expansion of the k^{th} partial sum, where $k \geq 3$. Observe that,

$$\sum_{n=0}^{k} q^{-n}=1+\frac{q^{k-1}+q^{k-2}+\ldots+q+1}{q^k}$$

$$=1+\frac{1}{\frac{q^k}{q^{k-1}+q^{k-2}+\ldots+q+1}}$$

$$=1+\cfrac{1}{\cfrac{q^k-1+1}{q^{k-1}+q^{k-2}+\ldots+q+1}}$$

$$=1+\cfrac{1}{\cfrac{(q-1)(q^{k-1}+q^{k-2}+\ldots+1)+1}{q^{k-1}+q^{k-2}+\ldots+q+1}}$$

(6) $$=1+\cfrac{1}{q-1+\cfrac{1}{q^{k-1}+q^{k-2}+\ldots+q+1}}.$$

Therefore, $1+\frac{1}{q}+\frac{1}{q^2}+\ldots.+\frac{1}{q^k}=[1,q-1,q^{k-1}+q^{k-2}+\ldots+q+1]$. Since

$\frac{1}{q^{k-1}+q^{k-2}+\ldots+q+1}\to 0$ as $k\to\infty$, from (6) we see that the simple geometric series

$\sum_{n=0}^{\infty}q^{-n}$ converges to $1+\frac{1}{q-1}=\frac{q}{q-1}$, as expected.

It is useful to note that in example 6 we used a quite obvious but useful fact that in general,

$\lim_{m\to\infty}\left([a_0,a_1,\ldots,a_k,m]\right)=[a_0,a_1,\ldots,a_k]$.

In the next example, we consider the expansion of the series $\frac{p}{q}+\frac{1}{q^2}+\frac{1}{q^3}+\ldots=\frac{p}{q}+\sum_{n=2}^{\infty}q^{-n}$, for specific values of *p* and *q*. Although the expansion of this series has similarities to the one above, the $\frac{p}{q}$ term complicates things.

<u>Example 7</u> Suppose $\frac{p}{q}=\frac{129}{412}=[0,3,5,6,4]$. The following table gives the first 5 partial sums of the series $\frac{p}{q}+\sum_{n=2}^{\infty}q^{-n}$.

n	Continued Fraction Expansion
2	[0, 3, 5, 6, 5, 3, 6, 5, 3]
3	[0, 3, 5, 6, 5, 3, 8, 3, 15, 1, 2, 1, 1, 4, 3]
4	[0, 3, 5, 6, 5, 3, 8, 3, 6607, 1, 2, 1, 1, 4, 3]
5	[0, 3, 5, 6, 5, 3, 8, 3, 2722511, 1, 2, 1, 1, 4, 3]
6	[0, 3, 5, 6, 5, 3, 8, 3, 1121674959, 1, 2, 1, 1, 4, 3]

Table 2

We see that the expansion becomes fixed except for one partial quotient when $n \geq 3$. The non-fixed partial quotients are 15, 6607, 2722511, and 1121674959. Observe that:

$$\begin{aligned}15 &= 16(1) - 1,\\ 6607 &= 16(412 + 1) - 1,\\ 2722511 &= 16(412^2 + 412 + 1) - 1,\\ 1121674959 &= 16(412^3 + 412^2 + 412 + 1) - 1.\end{aligned}$$

From this we see that the non-fixed partial quotient corresponding to the n^{th} partial sum($n \geq 3$) takes the form $16\sum_{k=0}^{n-3}\left(q^k\right) - 1$. We can use the information from Table 2 to tell us what the series $\frac{p}{q} + \sum_{n=2}^{\infty} q^{-n}$ converges to. Similar to example 6, the non-fixed partial quotients go to infinity as n goes to infinity. Thus, $\frac{p}{q} + \sum_{n=2}^{\infty} q^{-n}$ converges to $[0,3,5,6,5,3,8,3] = \frac{13255}{42333}$. As mentioned prior to example 7, the expansions that appear in Table 2 are much more difficult to predict than the expansions that appear in the partial sums of a geometric series. However based on several examples, we were able to make some conjectures that predict certain patterns that will appear. We make no attempt to prove these in this paper, but they appear in Chapter 5.

In example 4, we generated a table of continued fraction expansions for numbers of the form $1 + 2^{-2} + 2^{-9} + 2^{-i}$ for $10 \leq i \leq 22$. We noted a particular pattern that occurred for values of $i \geq 19$ where only a single partial quotient changed. Part of the table is given again below.

i	Continued Fraction Expansion
18	[1, 3, 1, 31, 5, 3, 31, 1, 3]
19	[1, 3, 1, 31, 4, 1, 1, 3, 31, 1, 3]
20	[1, 3, 1, 31, 4, 3, 1, 3, 31, 1, 3]
21	[1, 3, 1, 31, 4, 7, 1, 3, 31, 1, 3]
22	[1, 3, 1, 31, 4, 15, 1, 3, 31, 1, 3]

Table 3

An inspection of the table reveals that the pattern of interest occurs at the next integer after $i = 18$, the square of the denominator of our original number, $1 + 2^{-2} + 2^{-9}$. A clear pattern is also evident in the only non-fixed quotients of the expansions, i.e. 1, 3, 7, and 15. They are all of the form $2^k - 1$ where *k* is the difference between *i* and 18. This is not true in general. That is, if we constructed a similar table with a different initial number, the non-fixed quotients may not have the form $2^k - 1$. However, the least common denominator of our original number, *x*, is 2^9 and so its square is 2^{18}. When $i = 21$ for instance, the denominator of the resulting number is $2^3(2^{18})$ which is 8 multiples of 2^{18}, while the non-fixed partial quotient is $8 - 1 = 7$. This is actually the key to the value of the non-fixed term in general as we see in the following theorem.

Theorem 10 Suppose $\frac{p}{q} = [a_o, a_1, a_2, ..., a_n]$ where $n \geq 1$ and $a_n \neq 1$. Then for $k \in \mathbb{Z}^+$,

$$\frac{p}{q} + \frac{(-1)^n}{(k+1)q^2} = [a_o, a_1, a_2, ..., a_{n-1}, a_n, k, 1, a_n - 1, a_{n-1}, a_{n-2}, ..., a_1]$$

and

$$\frac{p}{q} - \frac{(-1)^n}{(k+1)q^2} = [a_o, a_1, a_2, ..., a_{n-1}, a_n - 1, 1, k, a_n, a_{n-1}, a_{n-2}, ..., a_1].$$

Proof: Since continued fraction expansions are unique, consider the following cases.

Case 1: Suppose $x=[a_o, a_1, a_2, \ldots a_{n-1}, a_n, k, 1, a_n - 1, a_{n-1}, \ldots a_2, a_1]$, we will show that x necessarily can be written in the form $\frac{p}{q}+\frac{(-1)^n}{(k+1)q^2}$. Observe that the first $n+1$ convergents of $\frac{p}{q}$ and x are exactly the same, so $\frac{p_i}{q_i}$ for $i \le n$ are all convergents of x. Now suppose $x=[a_o, a_1, a_2, \ldots a_{n-1}, a_n, \alpha]$ where

$$\alpha=[k,1,a_n-1,a_{n-1},\ldots,a_2,a_1]=k+\cfrac{1}{1+\cfrac{1}{a_n-1+\cfrac{1}{a_{n-1}+\cfrac{1}{\ddots+\cfrac{1}{a_1}}}}}$$

$$=k+\cfrac{1}{1+\cfrac{1}{-1+\left(a_n+\cfrac{1}{a_{n-1}+\cfrac{1}{\ddots+\cfrac{1}{a_1}}}\right)}}.$$

Since $\frac{p}{q}=[a_o, a_1, a_2, \ldots, a_n]$, by Theorem 3, $\frac{q_n}{q_{n-1}}=[a_n, \ldots a_2, a_1]$. Hence,

$$\alpha=k+\cfrac{1}{1+\cfrac{1}{-1+\cfrac{q_n}{q_{n-1}}}}=k+\cfrac{1}{1+\cfrac{q_{n-1}}{q_n-q_{n-1}}}$$

$$=k+\frac{q_n-q_{n-1}}{q_n}=\frac{kq_n+q_n-q_{n-1}}{q_n}.$$

Now since $x=\frac{\alpha p_n+p_{n-1}}{\alpha q_n+q_{n-1}}$, by substitution we get the following:

$$x=\frac{\alpha p_n+p_{n-1}}{\alpha q_n+q_{n-1}}=\frac{(\frac{kq_n+q_n-q_{n-1}}{q_n})p_n+p_{n-1}}{(\frac{kq_n+q_n-q_{n-1}}{q_n})q_n+q_{n-1}}$$

$$=\frac{kp_nq_n+p_nq_n+(p_{n-1}q_n-p_nq_{n-1})}{q_n(kq_n+q_n)}=\frac{kpq+pq+(-1)^n}{q(kq+q)}\text{ By Theorem 2 (ii)}$$

$$=\frac{pq(k+1)}{q^2(k+1)}+\frac{(-1)^n}{q^2(k+1)}=\frac{p}{q}+\frac{(-1)^n}{(k+1)q^2}.$$

<u>*Case 2*:</u> Now let $y=[a_o,a_1,a_2,...a_{n-1},a_n-1,1,k,a_n,a_{n-1},...a_2,a_1]$. We now show that y can be written in the form $\frac{p}{q}-\frac{(-1)^n}{(k+1)q^2}$. This time let the convergents of $\frac{p}{q}$ be denoted by $\frac{p_i}{q_i}$ for $i\le n$ while we denote the convergents of y by $\frac{p_i^{'}}{q_i^{'}}$. Notice that from the expansion of y ,

$\frac{p_i}{q_i}=\frac{p_i^{'}}{q_i^{'}}$ for $i\le n-1$, $p_n\ne p_n^{'}$ and $q_n\ne q_n^{'}$, and $p_{n+1}^{'}=p_n$ and $q_{n+1}^{'}=q_n$. We can easily calculate $p_n^{'}$ and $q_n^{'}$: $p_n^{'}=(a_n-1)p_{n-1}+p_{n-2}=a_np_{n-1}+p_{n-2}-p_{n-1}=p_n-p_{n-1}$ and similarly, $q_n^{'}=q_n-q_{n-1}$. Now again let's suppose $y=[a_o,a_1,a_2,...a_{n-1},a_n-1,1,\beta]$ where $\beta=[k,a_n,a_{n-1},...a_2,a_1]$. Then, again using Theorem 3 we have:

$$k+\cfrac{1}{a_n+\cfrac{1}{a_{n-1}+\cfrac{1}{a_{n-2}+\cfrac{1}{\ddots+\cfrac{1}{a_1}}}}}=k+\frac{1}{\frac{q_n}{q_{n-1}}}$$

$$=k+\frac{q_{n-1}}{q_n},\quad\text{which implies}$$

$$\beta = \frac{kq_n + q_{n-1}}{q_n}.$$

Once again we use substitution into the equation $y = \frac{\beta p'_{n+1} + p'_n}{\beta q'_{n+1} + q'_n} = \frac{\beta p_n + p_n - p_{n-1}}{\beta q_n + q_n - q_{n-1}}$ and we get:

$$y = \frac{\beta p_n + p_n - p_{n-1}}{\beta q_n + q_n - q_{n-1}} = \frac{\left(\frac{kq_n + q_{n-1}}{q_n}\right) p_n + p_n - p_{n-1}}{\left(\frac{kq_n + q_{n-1}}{q_n}\right) q_n + q_n - q_{n-1}}$$

$$= \frac{kpq + pq + (p_n q_{n-1} - p_{n-1} q_n)}{kq^2 + q^2} = \frac{kpq + pq + (-1)^{n-1}}{(k+1)q^2} \text{ By Theorem 2 (ii)}$$

$$= \frac{kpq + pq + (-1)^{n-1}}{q^2(k+1)} = \frac{p}{q} + \frac{(-1)^{n-1}}{(k+1)q^2}.$$

□

From the proof of Theorem 10, we see that the expansion changes based on whether *n* is even or odd. The following corollaries tell us exactly how each case plays out.

<u>Corollary</u> When *n* is even,

$$\frac{p}{q} + \frac{1}{(k+1)q^2} = [a_o, a_1, a_2, ..., a_{n-1}, a_n, k, 1, a_n - 1, a_{n-1}, a_{n-2}, ..., a_1]$$

$$\frac{p}{q} - \frac{1}{(k+1)q^2} = [a_o, a_1, a_2, ..., a_{n-1}, a_n - 1, 1, k, a_n, a_{n-1}, a_{n-2}, ..., a_1].$$

<u>Corollary</u> When *n* is odd,

$$\frac{p}{q} + \frac{1}{(k+1)q^2} = [a_o, a_1, a_2, ..., a_{n-1}, a_n - 1, 1, k, a_n, a_{n-1}, a_{n-2}, ..., a_1]$$

$$\frac{p}{q}-\frac{1}{(k+1)q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n,k,1,a_n-1,a_{n-1},a_{n-2},...,a_1].$$

Theorem 10 tells how the expansions of rational numbers change as we add and subtract integral multiples of q^2. However, it does not tell us what happens when that multiple is 1. The next theorem takes care of that.

<u>Theorem 11</u> Suppose $\frac{p}{q}=[a_0,a_1,...,a_n]$ where $n\geq 1$, $q\neq 0$,and $a_n\neq 1$. Then,

$$\frac{p}{q}+\frac{(-1)^n}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},...,a_1]$$

and

$$\frac{p}{q}-\frac{(-1)^n}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n-1,a_n+1,a_{n-1},a_{n-2},...,a_1].$$

<u>Proof:</u> We proceed in the same manner as in the proof of Theorem 10.

Case 1: Suppose $x=[a_o,a_1,a_2,...,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},...,a_1]$. We then note that the convergents of x and $\frac{p}{q}$ are the same up through $\frac{p_{n-1}}{q_{n-1}}$. Let's denote the n^{th} convergent of x by $\frac{p_n'}{q_n'}$. Then we see from Theorem 2 (i) that $p_n'=(a_n+1)p_{n-1}+p_{n-2}=p_n+p_{n-1}$. Similarly, $q_n'=q_n+q_{n-1}$. Now if $x=[a_0,a_1,...,a_{n-1},a_n+1,\alpha]$ where $\alpha=[a_n-1,a_{n-1},...,a_1]$ then

$$\alpha=-1+a_n+\cfrac{1}{a_{n-1}+\cfrac{1}{\ddots+\cfrac{1}{a_1}}}$$

$$=-1+\frac{q_n}{q_{n-1}} \qquad \text{by Theorem 3}$$

$$= \frac{q_n - q_{n-1}}{q_{n-1}}.$$

By Theorem 2 (iii), $x = \frac{\alpha p_n' + p_{n-1}}{\alpha q_n' + q_{n-1}} = \frac{\alpha(p_n + p_{n-1}) + p_{n-1}}{\alpha(q_n + q_{n-1}) + q_{n-1}}$ and substituting $\alpha = \frac{q_n - q_{n-1}}{q_{n-1}}$ we get:

$$x = \frac{\alpha(p_n + p_{n-1}) + p_{n-1}}{\alpha(q_n + q_{n-1}) + q_{n-1}} = \frac{\left(\frac{q_n - q_{n-1}}{q_{n-1}}\right)(p_n + p_{n-1}) + p_{n-1}}{\left(\frac{q_n - q_{n-1}}{q_{n-1}}\right)(q_n + q_{n-1}) + q_{n-1}} \bullet \frac{q_{n-1}}{q_{n-1}}$$

$$= \frac{(q_n - q_{n-1})(p_n + p_{n-1}) + p_{n-1}q_{n-1}}{(q_n - q_{n-1})(q_n + q_{n-1}) + q_{n-1}^2}$$

$$= \frac{p_n q_n + (p_{n-1}q_n - p_n q_{n-1})}{q_n^2} = \frac{p_n q_n + (-1)^n}{q_n^2} \quad \text{by Theorem 2 (ii)}$$

$$= \frac{p_n q_n + (-1)^n}{q_n^2} = \frac{p}{q} + \frac{(-1)^n}{q^2}.$$

Case 2: Suppose $y = [a_o, a_1, a_2, ..., a_{n-1}, a_n - 1, \beta]$ where $\beta = [a_n + 1, a_{n-1}, a_{n-2}, ..., a_1]$. This time,

$p_n' = (a_n - 1)p_{n-1} + p_{n-2} = p_n - p_{n-1}$ and so $q_n' = q_n - q_{n-1}$. Thus, by Theorems 2 and 3,

$$\beta = a_n + 1 + \cfrac{1}{a_{n-1} + \cfrac{}{\ddots + \cfrac{1}{a_1}}} = 1 + \frac{q_n}{q_{n-1}} = \frac{q_n + q_{n-1}}{q_{n-1}}, \text{ and so}$$

$$y = \frac{\beta p_n' + p_{n-1}}{\beta q_n' + q_{n-1}} = \frac{\beta(p_n - p_{n-1}) + p_{n-1}}{\beta(q_n - q_{n-1}) + q_{n-1}}. \text{ Substitution then yields,}$$

$$y = \frac{\left(\frac{q_n + q_{n-1}}{q_{n-1}}\right)(p_n - p_{n-1}) + p_{n-1}}{\left(\frac{q_n + q_{n-1}}{q_{n-1}}\right)(q_n - q_{n-1}) + q_{n-1}} \bullet \frac{q_{n-1}}{q_{n-1}}$$

$$=\frac{(q_n+q_{n-1})(p_n-p_{n-1})+p_{n-1}q_{n-1}}{(q_n+q_{n-1})(q_n-q_{n-1})+q_{n-1}^2}=\frac{p_nq_n+(p_nq_{n-1}-p_{n-1}q_n)}{q_n^2}$$

$$=\frac{p_nq_n+(-1)^{n+1}}{q_n^2} \qquad \text{By Theorem 2 (ii)}$$

$$=\frac{p}{q}+\frac{(-1)^{n+1}}{q^2}.$$

$\square$

<u>Corollary</u> When n is even,

$$\frac{p}{q}+\frac{1}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},...,a_1],$$

$$\frac{p}{q}-\frac{1}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n-1,a_n+1,a_{n-1},a_{n-2},...,a_1].$$

<u>Corollary</u> When n is odd,

$$\frac{p}{q}+\frac{1}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n-1,a_n+1,a_{n-1},a_{n-2},...,a_1],$$

$$\frac{p}{q}-\frac{1}{q^2}=[a_o,a_1,a_2,...,a_{n-1},a_n+1,a_n-1,a_{n-1},a_{n-2},...,a_1].$$

To help clarify Theorems 10 and 11 let's consider some examples.

<u>Example 8</u> Suppose $\frac{p}{q}=\frac{729}{557}=[1,3,4,5,7,1]=[1,3,4,5,8]$. Since n is even, from Theorem 11 we can conclude without any computation that

$$\frac{729}{557}+\frac{1}{557^2}=[1,3,4,5,9,7,5,4,3].$$

Again, we don't need to do any computation to see that from Theorem 10 when $k = 272$,

$$\frac{729}{557} + \frac{1}{(272+1)557^2} = [1,3,4,5,8,272,1,7,5,4,3].$$

So far in this paper we have started with the continued fraction expansion of $\frac{p}{q}$ and then observed how the continued fraction expansion of $\frac{p}{q} + r$ changed as we picked various values for *r*. However, sometimes we start with a certain form of continued fraction expansion, and find out what value of *r* would correspond to this expansion. For instance, looking at Theorems 10 and 11, we see that the expansions given are very close to being a palindrome. A palindrome is a word, phrase, or number that is the same read backwards and forwards, such as "12321" or "wow". These expansions lead to the following question. If we ignore the first partial quotient, for what value of *r* does the expression $\frac{p}{q} + r$ have a continued fraction expansion which is a palindrome? That is, an expansion of the form $[a_0, a_1, ..., a_{n-1}, a_n, a_n, a_{n-1}, ...a_1]$, $[a_0, a_1, ..., a_{n-1}, a_n, a_{n-1}, ...a_1]$, or $[a_0, a_1, ..., a_{n-1}, a_n, k, a_n, a_{n-1}, ...a_1]$ for some positive integer *k*. This question is not difficult to answer using the same technique used for the proofs of Theorem 10 and 11. It turns out that to find this value of *r* we need to know what q_{n-1} is.

<u>Theorem 12</u> Suppose $\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_n]$ where $n \geq 1$. Then

$$\frac{p_n}{q_n} + \frac{(-1)^n}{q_n(kq_n + 2q_{n-1})} = [a_0, a_1, ..., a_{n-1}, a_n, k, a_n, a_{n-1}, ..., a_1].$$

<u>Proof</u> Let $x = [a_0, a_1, ..., a_{n-1}, a_n, x_{n+1}]$ where $x_{n+1} = [k, a_n, a_{n-1}, ..., a_1]$. Then,

$$x_{n+1} = k + \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}} = k + \frac{1}{\frac{q_n}{q_{n-1}}} = \frac{kq_n + q_{n-1}}{q_n} \quad \text{by Theorem 3.}$$

Now by Theorem 2(iii) and substitution,

$$x=\frac{x_{n+1}p_n+p_{n-1}}{x_{n+1}q_n+q_{n-1}}=\frac{\left(\frac{kq_n+q_{n-1}}{q_n}\right)p_n+p_{n-1}}{\left(\frac{kq_n+q_{n-1}}{q_n}\right)q_n+q_{n-1}}=\frac{p_n(kq_n+q_{n-1})+p_{n-1}q_n}{q_n(kq_n+q_{n-1})+q_nq_{n-1}}=\frac{kp_nq_n+p_nq_{n-1}+p_{n-1}q_n}{q_n(kq_n+2q_{n-1})}$$

.

Now observe that

$$x-\frac{p_n}{q_n}=\frac{kp_nq_n+p_nq_{n-1}+p_{n-1}q_n}{q_n(kq_n+2q_{n-1})}-\frac{p_n}{q_n}$$

$$=\frac{kp_nq_n+p_nq_{n-1}+p_{n-1}q_n-p_n(kq_n+2q_{n-1})}{q_n(kq_n+2q_{n-1})}$$

$$=\frac{kp_nq_n+p_nq_{n-1}+p_{n-1}q_n-kp_nq_n-2p_nq_{n-1}}{q_n(kq_n+2q_{n-1})}$$

$$=\frac{p_{n-1}q_n-p_nq_{n-1}}{q_n(kq_n+2q_{n-1})}=\frac{(-1)^n}{q_n(kq_n+2q_{n-1})} \quad \text{by Theorem 2(ii).}$$

Therefore, $\frac{p_n}{q_n}+\frac{(-1)^n}{q_n(kq_n+2q_{n-1})}=[a_0,a_1,...,a_{n-1},a_n,k,a_n,a_{n-1},...,a_1]$ as desired.

□

Theorem 13 Suppose $\frac{p_n}{q_n}=[a_0,a_1,a_2,...,a_n]$ where $n\geq 1$. Then

$$\frac{p_n}{q_n}+\frac{(-1)^n q_{n-1}}{q_n(q_n^2+q_{n-1}^2)}=[a_0,a_1,...,a_{n-1},a_n,a_n,a_{n-1},...,a_1].$$

Proof Once again let $x=[a_0,a_1,...,a_{n-1},a_n,x_{n+1}]$ where $x_{n+1}=[a_n,a_{n-1},...,a_1]$. Then by Theorem 3, $x_{n+1}=\frac{q_n}{q_{n-1}}$. By Theorem 2(iii),

$$x=\frac{x_{n+1}p_n+p_{n-1}}{x_{n+1}q_n+q_{n-1}}=\frac{\left(\frac{q_n}{q_{n-1}}\right)p_n+p_{n-1}}{\left(\frac{q_n}{q_{n-1}}\right)q_n+q_{n-1}}=\frac{p_nq_n+p_{n-1}q_{n-1}}{q_n^2+q_{n-1}^2}$$ so that

$$x-\frac{p_n}{q_n}=\frac{p_nq_n+p_{n-1}q_{n-1}}{q_n^2+q_{n-1}^2}-\frac{p_n}{q_n}=\frac{(p_nq_n+p_{n-1}q_{n-1})q_n-p_n(q_n^2+q_{n-1}^2)}{q_n(q_n^2+q_{n-1}^2)}$$

$$=\frac{p_{n-1}q_{n-1}q_n-p_nq_{n-1}^2}{q_n(q_n^2+q_{n-1}^2)}$$

$$=\frac{q_{n-1}(p_{n-1}q_n-p_nq_{n-1})}{q_n(q_n^2+q_{n-1}^2)}$$

$$=\frac{q_{n-1}(-1)^n}{q_n(q_n^2+q_{n-1}^2)}$$ by Theorem 2(ii).

We now see that $\frac{p_n}{q_n}+\frac{q_{n-1}(-1)^n}{q_n(q_n^2+q_{n-1}^2)}=[a_0,a_1,...,a_{n-1},a_n,a_n,a_{n-1},...,a_1]$.

□

<u>Theorem 14</u> Suppose $\frac{p_n}{q_n}=[a_0,a_1,a_2,...,a_n]$ where $n\geq 2$. Then

$$\frac{p_n}{q_n}+\frac{(-1)^n q_{n-2}}{q_nq_{n-1}(q_n+q_{n-2})}=[a_0,a_1,...,a_{n-1},a_n,a_{n-1},...,a_1].$$

<u>Proof</u> Let $x=[a_0,a_1,...,a_{n-1},a_n,x_{n+1}]$ so that $x_{n+1}=[a_{n-1},a_{n-2},...,a_1]=\frac{q_{n-1}}{q_{n-2}}$ by Theorem 3.

Then by Theorem 2(iii) we have,

$$x = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} = \frac{\left(\frac{q_{n-1}}{q_{n-2}}\right)p_n + p_{n-1}}{\left(\frac{q_{n-1}}{q_{n-2}}\right)q_n + q_{n-1}} = \frac{p_n q_{n-1} + p_{n-1}q_{n-2}}{q_n q_{n-1} + q_{n-1}q_{n-2}}.$$

Thus,

$$x - \frac{p_n}{q_n} = \frac{p_n q_{n-1} + p_{n-1}q_{n-2}}{q_n q_{n-1} + q_{n-1}q_{n-2}} - \frac{p_n}{q_n}$$

$$= \frac{q_n(p_n q_{n-1} + p_{n-1}q_{n-2}) - p_n(q_n q_{n-1} + q_{n-1}q_{n-2})}{q_n q_{n-1}(q_n + q_{n-2})}$$

$$= \frac{p_{n-1}q_n q_{n-2} - p_n q_{n-1}q_{n-2}}{q_n q_{n-1}(q_n + q_{n-2})} = \frac{q_{n-2}(p_{n-1}q_n - p_n q_{n-1})}{q_n q_{n-1}(q_n + q_{n-2})}$$

$$= \frac{(-1)^n q_{n-2}}{q_n q_{n-1}(q_n + q_{n-2})} \qquad \text{by Theorem 2(ii).}$$

Therefore, $\frac{p_n}{q_n} + \frac{(-1)^n q_{n-2}}{q_n q_{n-1}(q_n + q_{n-2})} = [a_0, a_1, ..., a_{n-1}, a_n, a_{n-1}, ..., a_1]$ as desired.

□

Although Theorems 12, 13, and 14 have continued fraction expansions that are more pleasing to the eye than the expansions given in Theorems 10 and 11, the next example shows that they are not as natural. In order to apply Theorems 12, 13, and 14, we need to know the values of q_{n-1} and sometimes q_{n-2}.

Example 9 Notice that $\frac{2222}{643} = [3,2,5,7,8] = [3,2,5,7,7,1]$. To proceed, we must first pick one of these expansions that we desire to work with. Let's work with $[3,2,5,7,7,1]$ this time. We start by observing that since $a_5 = 1$ is the last term, $n = 5$. Next we find $q_{5-1} = q_4$. The convergents of $\frac{2222}{643}$ are $\frac{3}{1}, \frac{7}{2}, \frac{38}{11}, \frac{273}{79}, \frac{1949}{564}$, and $\frac{2222}{643}$ so $q_4 = 564$. Thus by Theorem 12 when $k = 32$ we see that

$$\frac{2222}{643} + \frac{(-1)^5}{643(32 \cdot 643 + 2 \cdot 564)} = [3,2,5,7,7,1,32,1,7,7,5,2].$$

By Theorem 13,

$$\frac{2222}{643} + \frac{(-1)^5 \cdot 564}{643(643^2 + 564^2)} = [3,2,5,7,7,1,1,7,7,5,2].$$

Finally by Theorem 14,

$$\frac{2222}{643} + \frac{(-1)^5 \cdot 79}{643 \cdot 564(643 + 79)} = [3,2,5,7,7,1,7,7,5,2].$$

There is actually a specific case where Theorems 12 and 13 can be thought of as being just as natural as Theorems 10 and 11. If $\frac{p}{q} = [0, a_1, ..., a_n]$ and $[a_1, ..., a_n]$ is a palindrome, then it follows that $q_{n-1} = p$. To see this, let $x = [a_1, a_2, ..., a_n]$ so that $\frac{p}{q} = 0 + \frac{1}{x}$, and thus $x = \frac{q}{p}$. Using this and the fact that x is a palindrome, Theorem 3 tells us $x = \frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, ..., a_1]$ and hence, $\frac{q}{q_{n-1}} = \frac{q}{p_n}$, giving $q_{n-1} = p$. Therefore we can replace each q_{n-1} with p_n thus eliminating the need to find the convergents to apply these theorems. In fact, one could accomplish this by first applying one of the Theorems 12, 13, or 14 to a rational number less than 1. From there, the numerator of the resulting rational number would serve as q_{n-1} illustrated in the next example.

Example 10 Suppose $x = \frac{4}{13} = [0,3,4]$. We first apply Theorem 13 so that the partial quotients after the first become a palindrome. Doing so gives $\frac{4}{13} + \frac{3}{13(13^2 + 3^2)} = [0,3,4,4,3] = \frac{55}{178}$.

Next, we will apply Theorem 12 with $k = 24$ to $\frac{p}{q} = \frac{55}{178}$. Since the continued fraction expansion of $\frac{55}{178}$ meets the conditions mentioned above, we know that $q_{n-1} = p$. Thus

$\frac{55}{178} + \frac{1}{178(24 \cdot 178 + 2 \cdot 55)} = [0,3,4,4,3,24,3,4,4,3]$, which again has an expansion with the form zero followed by a palindrome.

Theorems 10 and 11 have some interesting applications regarding infinite series. In particular, we will show that certain types of infinite series converge to an irrational number by observing that their continued fraction expansions are infinite. However, we will first want to define some tools to aid us in the proof.

We start by defining a function, *L*, on the real numbers. If $x \in \mathbb{R}$, then $L(x)$ is equal to the number partial quotients in the continued fraction expansion of x when the last partial quotient is not equal to one. We say $L(x) = \infty$ if x is irrational. That is, if $a_n \neq 1$,

$$L([a_0, a_1, a_2, ..., a_n]) = n+1$$

and

$$L([a_0, a_1, a_2, ...]) = \infty.$$

For example, if $x = [3,2,5,7,2] = [3,2,5,7,1,1]$ then $L(x) = 5$. If $y = \sqrt{2}$, then $L(y) = \infty$.

Next, we shall define the vector $\vec{V}$. Suppose $r = [a_o, a_1, a_2, ..., a_{n-1}, a_n]$, $a_n \neq 1$. Then $\vec{V}$ will represent all the partial quotients of *r* except the first two and the last. That is, $\vec{V} = [a_2, ..., a_{n-1}]$. Now let's denote the reverse of $\vec{V}$ by $\overrightarrow{V^R}$. That is, $\overrightarrow{V^R} = [a_{n-1}, a_{n-2}, ..., a_2]$. It is important to also note that $L(\vec{V}) = L(\overrightarrow{V^R}) = n-2$.

We will now use these tools to prove the following theorem.

Theorem 15 Let $\frac{p}{q}$ be a nonzero rational number. Then the infinite series $\frac{p}{q}+\sum_{i=1}^{\infty} q^{-2^i}$ converges to an irrational number. Furthermore, this irrational number is not a quadratic irrational.

Proof: Suppose $\frac{p}{q}=[a_o,a_1,a_2,...,a_n]$ where $a_n \neq 1$ and n is even (The case where n is odd is very similar). Now let $\overrightarrow{V_0}=(a_2,a_3,...,a_{n-1})$ so $\overrightarrow{V_0^R}=(a_{n-1},....,a_2)$ and thus $\frac{p}{q}=[a_o,a_1,\overrightarrow{V_0},a_n]$.

We know that $L\left(\overrightarrow{V_0}\right)=n-2=2^0 n-2$ and $L\left(\frac{p}{q}\right)=n+1=2^0 n+1$. By Theorem 11,

$$\frac{p}{q}+\frac{1}{q^2}=\frac{p}{q}+\sum_{i=1}^{1} q^{-2^i}=[a_o,a_1,\overrightarrow{V_0},a_n+1,a_n-1,\overrightarrow{V_0^R},a_1].$$

Now let $\overrightarrow{V_1}=\left(\overrightarrow{V_0},a_n+1,a_n-1,\overrightarrow{V_0^R}\right)$ so that $\frac{p}{q}+\sum_{i=1}^{1} q^{-2^i}=[a_0,a_1,\overrightarrow{V_1},a_1]$.

Note at this stage, if $a_1=1$ then the final partial quotient of $\frac{p}{q}+\sum_{i=1}^{1} q^{-2^i}$ would be converted to a_2+1. This would only force us to use the second Corollary of Theorem 10 instead of the first. The proof would be nearly identical.

Continuing on we see that,

$$L\left(\overrightarrow{V_1}\right)=2L\left(\overrightarrow{V_0}\right)+2=2(n-2)+2=2^1 n-2, \text{ and so}$$

$$L\left(\frac{p}{q}+\sum_{i=1}^{1} q^{-2^i}\right)=2n-2+3=2^1 n+1.$$

Once again by Theorem 11,

$$\frac{p}{q}+\sum_{i=1}^{2} q^{-2^i}=[a_0,a_1,\overrightarrow{V_1},a_1+1,a_1-1,\overrightarrow{V_1^R},a_1].$$

Define $\overrightarrow{V_2}=\left(\overrightarrow{V_1},a_1+1,a_1-1,\overrightarrow{V_1^R}\right)$ so that $\frac{p}{q}+\sum_{i=1}^{2} q^{-2^i}=[a_0,a_1,\overrightarrow{V_2},a_1]$. Hence,

$$L\left(\overrightarrow{V_2}\right)=2L\left(\overrightarrow{V_1}\right)+2=2(2n-2)+2=2^2n-2 \text{ and}$$

$$L\left(\frac{p}{q}+\sum_{i=1}^{2}q^{-2^i}\right)=2^2n-2+3=2^2n+1.$$

Suppose we continue to define $\overrightarrow{V_3},\overrightarrow{V_4},...,\overrightarrow{V_k}$ as we did above. That is, $\overrightarrow{V_0}=(a_2,a_3,...,a_{n-1})$, and $\overrightarrow{V_{j+1}}=\left(\overrightarrow{V_j},a_n+1,a_n-1,\overrightarrow{V_j^R}\right)$ for $j\geq 0$. We shall now prove by induction that for each $k\geq 0$ the following holds:

$$L\left(\overrightarrow{V_k}\right)=2^kn-2 \text{ and } L\left(\frac{p}{q}+\sum_{i=1}^{k}q^{-2^i}\right)=2^kn+1.$$

We have already shown that it holds for $k=0,1,2$. Suppose it holds for each of the integers from 0 up to some integer $m\geq 3$. Then $\overrightarrow{V_m}=\left(\overrightarrow{V_{m-1}},a_1+1,a_1-1,\overrightarrow{V_{m-1}^R}\right)$ and $\frac{p}{q}+\sum_{i=1}^{m}q^{-2^i}=[a_0,a_1,\overrightarrow{V_m},a_1]$. By our induction assumption,

$$L\left(\overrightarrow{V_m}\right)=2^mn-2 \text{ and } L\left(\frac{p}{q}+\sum_{i=1}^{m}q^{-2^i}\right)=2^mn+1.$$

Once again from Theorem 10 we know $\frac{p}{q}+\sum_{i=1}^{m+1}q^{-2^i}=[a_0,a_1,\overrightarrow{V_m},a_1+1,a_1-1,\overrightarrow{V_m^R},a_1]$. Since $\overrightarrow{V_{m+1}}=\left(\overrightarrow{V_m},a_1+1,a_1-1,\overrightarrow{V_m^R}\right)$ it follows that $\frac{p}{q}+\sum_{i=1}^{m+1}q^{-2^i}=[a_0,a_1,\overrightarrow{V_{m+1}},a_1]$. So,

$$L\left(\overrightarrow{V_{m+1}}\right)=2L\left(\overrightarrow{V_m}\right)+2=2(2^mn-2)+2=2^{m+1}n-2 \text{ and}$$

$$L\left(\frac{p}{q}+\sum_{i=1}^{m+1}q^{-2^i}\right)=2^{m+1}n-2+3=2^{m+1}n+1.$$

Hence, the result holds for each integer $k \geq 0$. This shows us that $L\left(\frac{p}{q}+\sum_{i=1}^{k} q^{-2^i}\right) \to \infty$ as $k \to \infty$. Therefore, we can conclude that $\frac{p}{q}+\sum_{i=1}^{\infty} q^{-2^i}$ converges to an irrational number by Theorem 4. Also, since this infinite continued fraction expansion is clearly not periodic, Theorem 6 tells us that the irrational number which the series converges to cannot be a quadratic irrational.

□

Using Theorems 10 and 11, we know the precise form of the expansion of $\frac{p}{q}+\frac{(-1)^n}{kq^2}$ for any positive integer *k*. In a similar manner to Theorem 12, we could use this knowledge to prove that any infinite series of the form $\frac{p}{q}+\sum_{i=1}^{\infty} q^{-b_i}$ where $\{b_i\}$ is any sequence with $b_{i+1} \geq 2b_i$ for all *i* converges to an irrational number.

To see Theorem 15 in action, consider the following example.

Example 11 Consider the series $\sum_{i=0}^{\infty} a^{-2^i} = \frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^4}+\ldots$ for some integer $a \geq 2$ Using Theorem 10, we can calculate the continued fraction expansions of the first several partial sums of this series without doing any computation. They are given below.

$$\sum_{i=0}^{0} a^{-2^i} = [0,a],\ \sum_{i=0}^{1} a^{-2^i} = [0,a-1,a+1],\ \sum_{i=0}^{2} a^{-2^i} = [0,a-1,a+2,a,a-1], \tag{7}$$

$$\sum_{i=0}^{3} a^{-2^i} = [0,a-1,a+2,a,a+1,a-1,a,a+2,a-1], \tag{8}$$

$$\sum_{i=0}^{4} a^{-2^i} = [0,a-1,a+2,a,a+1,a-1,a,a+2,a,a-2,a+2,a,a-1,a+1,a,a+2,a-1]. \tag{9}$$

From (7) we see that the first partial sum has two partial quotients and hence we apply the odd case of the corollary of Theorem 11. We then see that

$L\left(\sum_{i=0}^{1} a^{-2^i}\right)=3$ and $L\left(\sum_{i=0}^{2} a^{-2^i}\right)=3\times 2-1=5$ so we switch to the even case of the corollary and from then on we continue to use this case. (8) shows us that $L\left(\sum_{i=0}^{3} a^{-2^i}\right)=5\times 2-1=9$ and from (9) we see that $L\left(\sum_{i=0}^{4} a^{-2^i}\right)=9\times 2-1=17$. From these calculations, it is apparent that $L\left(\sum_{i=0}^{\infty} a^{-2^i}\right)=\infty$ and hence $\sum_{i=0}^{\infty} a^{-2^i}$ converges to an irrational number.

In the next example we consider the series $\sum_{i=0}^{\infty} 3^{-(2^{i+1}-1)} = \frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^7}+\frac{1}{3^{15}}+\ldots$ and show that it converges to an irrational number similar to the previous example. However, this time we apply Theorem 10 to find the expansion of the partial sums. This yields a more interesting pattern in the expansion.

Example 12 By applying Theorem 9 to the partial sums of the series $\sum_{i=0}^{\infty} 3^{-(2^{i+1}-1)}$ we get the following:

(10) $\sum_{i=0}^{0} 3^{-(2^{i+1}-1)} = [0,3],\ \sum_{i=0}^{1} 3^{-(2^{i+1}-1)} = [0,2,1,2,3],\ \sum_{i=0}^{2} 3^{-(2^{i+1}-1)} = [0,2,1,2,3,2,1,2,2,1,2]$

(11) $\sum_{i=0}^{3} 3^{-(2^{i+1}-1)} = [0,2,1,2,3,2,1,2,2,1,2,2,1,1,1,2,2,1,2,3,2,1,2],$

(12) $\sum_{i=0}^{4} 3^{-(2^{i+1}-1)} = [0,2,1,2,3,2,1,2,2,1,2,2,1,1,1,2,2,1,2,3,2,1,2,2,1,1,1,2,3,2,1,2,2,1,1,1,2,2,1,2,2,1,2,3,2,1,2]$

We first apply the odd case of Theorem 10 to get the second expansion in (10). Since $\frac{1}{3}+\frac{1}{3^3}=\frac{1}{3}+\frac{1}{(2+1)(3^2)}$ our value of *k* in Theorem 10 is 2. Note that in the calculation of each successive partial sum the value of *k* we are using is 2. For example let’s look at how we go from the second expansion to the third expansion in (10). The rational number represented by the third expansion is $\left(\frac{1}{3}+\frac{1}{3^3}\right)+\frac{1}{3^7}$. If we think of $\frac{p}{q}$ in Theorem 9 as $\left(\frac{1}{3}+\frac{1}{3^3}\right)$ then $q=3^3$ and hence we are adding $\frac{1}{(2+1)q^2}$ to it. It is not difficult to see that this is the case for each partial sum and hence why we get the expansions in (10), (11), and (12). Once again we see that the length of these expansions is clearly going to infinity so $\sum_{i=0}^{\infty} 3^{-(2^{i+1}-1)}$ represents an irrational number.

Let us once again refer back to the example 4. We started with the rational number $x=1+2^{-2}+2^{-9}$ which has expansion $[1,3,1,31,4]$. Observe that in Table 1, only once *i* was greater than 18 did each expansion start with $[1,3,1,31,4,...]$, the entire expansion of x. The next few theorems answer the questions of exactly what value needs to be added to x in order for this to occur. It turns out that the answer to this question depends on the n-1st convergent and once again the parity of *n*.

Theorem 16 If $x=[a_0,a_1,...,a_k,x_{k+1}]$ and we think of x as a function $f(x_{k+1})$, depending on x_{k+1} then on the interval $[1,\infty)$ we have

$$f(x_{k+1}) \text{ is } \begin{cases} A \ continuous \ monotonically \ decreasing \ function \ when \ k \ is \ even \\ A \ continuous \ monotonically \ increasing \ function \ when \ k \ is \ odd. \end{cases}$$

Proof: We have,

$$F'(x_{k+1})=\frac{(x_{k+1}q_k+q_{k-1})p_k-(x_{k+1}p_k+p_{k-1})q_k}{(x_{k+1}q_k+q_{k-1})^2}$$

$$=\frac{p_kq_{k-1}-p_{k-1}q_k}{(x_{k+1}q_k+q_{k-1})^2}$$

(13) $$=\frac{(-1)^{k+1}}{(x_{k+1}q_k+q_{k-1})^2}$$ by Theorem 2 (ii).

From (13) it is now clear that when k is even F' is always negative and when k is odd F' is always positive, thus the result follows.

$\square$

__Theorem 17__ Suppose $\frac{p_k}{q_k} = [a_0, a_1, ..., a_k]$. Then the interval on the real line with continued fraction expansion of the form $[a_0, a_1, ..., a_k, b_{k+1}, b_{k+2}, ...]$ where b_i is a positive integer for each i is:

$$\begin{cases} \left(\frac{p_k}{q_k}, \frac{p_k + p_{k-1}}{q_k + q_{k-1}}\right] \text{ if k is even} \\ \left[\frac{p_k + p_{k-1}}{q_k + q_{k-1}}, \frac{p_k}{q_k}\right) \text{if k is odd.} \end{cases}$$

__Proof:__ Suppose $f(x) = [a_0, a_1, ..., a_k, x]$ where $x = [b_{k+1}, b_{k+2}, ...]$. Then f has domain $[1, \infty)$ and by Theorem 13 f is an increasing function when k is odd and a decreasing function when k is even. By Theorem 2(iii),

$$f(x) = \frac{xp_k + p_{k-1}}{xq_k + q_{k-1}}.$$

Since

$$f(1) = \frac{p_k + p_{k-1}}{q_k + q_{k-1}} \text{ and } \lim_{x \to \infty} f(x) = \frac{p_k}{q_k},$$

the result follows.

$\square$

Note that in Theorem 17, this is one of the rare cases where we don't put the restriction that $a_k \neq 1$. So one has to choose carefully if the desired form of the expansion has $a_k = 1$ or not. Since $[a_0, a_1, ..., a_{k-1}, 1]$ is equal to but has one more convergent and partial quotient than

$[a_0, a_1, ..., a_{k-1} + 1]$; it must be understood what the parity of k is and what the values of the convergents $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{p_k}{q_k}$ are, as they are different depending on what your desired form is.

Making use of Theorem 17 leads us the to the following theorem, which now answers the question of what quantities we need to add to a rational number in order to preserve all or most of its partial quotients.

<u>Theorem 18</u> Suppose $\frac{p_n}{q_n} = [a_0, a_1, ..., a_{n-1}, a_n]$ then $\frac{p_n}{q_n} + (-1)^n \cdot r$ where $0 \le r \le \frac{1}{q_n(q_n + q_{n-1})}$ has a continued fraction expansion of the form $[a_0, a_1, ..., a_{n-1}, a_n, b_{n+1}, b_{n+2}, ...]$.

<u>Proof:</u> From Theorem 17, we know that in order for the expansion of $\frac{p_n}{q_n} + (-1)^n \cdot r$ to have the desired form, we need

(14)
$$\frac{p_n}{q_n} + (-1)^n \cdot r \in \begin{cases} \left(\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right] & \text{if n is even} \\ \left[\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n} \right) & \text{if n is odd.} \end{cases}$$

Observe that

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_n q_{n-1}}{q_n(q_n + q_{n-1})} = \frac{(-1)^n}{q_n(q_n + q_{n-1})}, \qquad \text{By Theorem 2(ii).}$$

Hence,

(15)
$$\frac{p_n}{q_n} + \frac{(-1)^n}{q_n(q_n + q_{n-1})} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Since $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ is the upper or lower bound on the intervals above, we see from (14) and (15) that any r such that $0 \le r \le \frac{1}{q_n(q_n + q_{n-1})}$ will have the desired expansion form.

$\square$

In investigating quantities of the form $\frac{p}{q}+r$ for small values of *r* given that $\frac{p}{q}=[a_0,a_1,...,a_n]$ we can see from (14) that when *n* is odd, adding any positive number *r*, no matter how small, will result in an expansion that does not preserve every partial quotient of $\frac{p}{q}$. However, recall that any rational number has precisely two expansions. That is, if $a_n \neq 1$ then $[a_0,a_1,...,a_n]=[a_0,a_1,...,a_n-1,1]$. With a proper re-indexing this now changes the parity of *n* and adding small enough quantities to this will now preserve every partial quotient. From this comes about the following corollary to Theorem 17.

<u>Corollary</u> Suppose $\frac{p_n}{q_n}=[a_0,a_1,...,a_{n-1},a_n]$ where $a_n \neq 1$. The interval on the real line with continued fraction expansion of the form $[a_0,a_1,...,a_{n-1},a_n-1,1,b_{n+2},b_{n+3},...]$ is

$$\left(\frac{p_n}{q_n},\frac{2p_n-p_{n-1}}{2q_n-q_{n-1}}\right] \text{ when n is odd}$$
$$\left[\frac{2p_n-p_{n-1}}{2q_n-q_{n-1}},\frac{p_n}{q_n}\right) \text{ when n is even.}$$

Also, $\frac{p_n}{q_n}+(-1)^{n+1}\cdot s$ where $0 \le s \le \frac{1}{q_n(2q_n-q_{n-1})}$ will have a continued fraction expansion of the form

$$[a_0,a_1,...,a_{n-1},a_n-1,1,b_{n+2},b_{n+3},...].$$

We can get another nice result by observing that since $q_n > q_{n-1}$, it follows that $\frac{1}{2q_n^2} < \frac{1}{q_n(q_n+q_{n-1})}$. We use this to give a corollary to Theorem 18.

Corollary Suppose $\frac{p}{q}=[a_0,a_1,...,a_n]$. Now pick $a_n=1$, if necessary, so that *n* is even. Then,

$\frac{p}{q}+r=[a_0,a_1,...,a_n,b_{n+1},...]$ if $0\le r\le \frac{1}{2q^2}$. If instead we choose the expansion with an odd

n, then $\frac{p}{q}+r=[a_0,a_1,...,a_n,b_{n+1},...]$ if $\frac{-1}{2q^2}\le r\le 0$.

Notice that in Theorem 18 we started with a rational number $\frac{p}{q}$ and gave the real number r so that $\frac{p}{q}+(-1)^n r$ had a continued fraction expansion which started with the entire continued fraction expansion of $\frac{p}{q}$. This begs the question of what happens if we replace $\frac{p}{q}$ with some irrational number x. In this case x has an infinite continued fraction expansion so the expansion of $x+r$ clearly cannot begin with the entire expansion of x. We instead ask for what values of r will the continued fraction expansions of x and $x+r$ have the same first *n* partial quotients? That is, for what values of r does $x=[a_0,a_1,...,a_n,a_{n+1},a_{n+2},...]$ and $x+r=[a_0,a_1,...,a_n,b_{n+1},b_{n+2},...]$? To find a precise answer we can use Theorem 17, however, the Theorem below gives a nicer result.

Theorem 19 Suppose the irrational number x has continued fraction expansion of the form $[a_0,a_1,...,a_{n-1},a_n,a_{n+1},...]$ where $a_{n+1}>1$. Then the continued fraction expansion of $x+(-1)^n\cdot r$ where $r<\frac{1}{2q_n^2+3q_nq_{n-1}+q_{n-1}^2}$ will have an expansion of the form $[a_0,a_1,...,a_{n-1},a_n,b_{n+1},...]$.

Proof: We prove the case where *n* is even. According to Theorem 17 we know that, since *n* is even, we need to prove that $x+r\in\left(\frac{p_n}{q_n},\frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right)$. Let $x=[a_0,a_1,a_2,...,a_n,x_{n+1}]$ so by Theorem 2(iii) $x=\frac{x_{n+1}p_n+p_{n-1}}{x_{n+1}q_n+q_{n-1}}$. It is clear that $x+r>\frac{p_n}{q_n}$ so we just need to show that

$$x+r<\frac{p_n+p_{n-1}}{q_n+q_{n-1}},\text{ or } r<\frac{p_n+p_{n-1}}{q_n+q_{n-1}}-\frac{x_{n+1}p_n+p_{n-1}}{x_{n+1}q_n+q_{n-1}}.$$

Observe that

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} = \frac{(p_nq_{n-1} - p_{n-1}q_n) + x_{n+1}(p_{n-1}q_n - p_nq_{n-1})}{(x_{n+1}q_n + q_{n-1})(q_n + q_{n-1})}$$

$$= \frac{x_{n+1} - 1}{(x_{n+1}q_n + q_{n-1})(q_n + q_{n-1})} \qquad \text{by Theorem 2(ii).}$$

Now the function defined by $f(x_{n+1}) = \dfrac{x_{n+1} - 1}{(x_{n+1}q_n + q_{n-1})(q_n + q_{n-1})}$ is monotonically increasing on the interval $[1, \infty)$ and therefore takes on a minimum value at $f(1) = 0$. However, since we require $a_{n+1} > 1$ then $x_{n+1} > 2$. Then $f(2) = \dfrac{1}{2q_n^2 + 3q_n + q_{n-1}^2}$ and so clearly any *r* smaller will also work.

$\square$.

Corollary If *x* is an irrational number and has an expansion of the form $[a_0, a_1, ..., a_{n-1}, a_n, a_{n+1}, ...]$ where $a_{n+1} > 1$, then $x + (-1)^n \cdot r$ will have an expansion of the form $[a_0, a_1, ..., a_{n-1}, a_n, b_{n+1}, ...]$ if $r < \dfrac{1}{6q_n^2}$.

Example 13 Suppose $x = \dfrac{\sqrt{15} - 1}{2} = [1, 2, 3, 2, 3, 2, 3, ...] = [1, \overline{2, 3}]$ and we wish to add some quantity *r* so as to preserve the first 5 partial quotients of *x.* We find that $q_4 = 55$, and $q_3 = 16$, so $r = \dfrac{(-1)^4}{2 \cdot 55^2 + 3 \cdot 55 \cdot 16 + 16^2} = \dfrac{1}{8946}$. Now the first 7 partial quotients of $x + r$ are [1,2,3,2,3,1,11,…]. If we instead pick a slightly larger value of *r* such as $\dfrac{1}{2 \cdot 55^2 + \cdot 55 \cdot 16} = \dfrac{1}{6930}$ then the first 7 partial quotients of $x + r$ are [1,2,3,2,4,11,1,…]. Notice that this time the first 5 partial quotients of *r* are not preserved.

The result given in Theorem 19 is not optimal, since the optimal value for such an *r* to preserve the first *n* partial quotients would simply be found by using Theorem 17 and some algebra. However, this value depends on the value of the initial irrational number *x* while the value for *r* given in Theorem 19 only depends on q_n and q_{n-1}. It is clear that we can find even smaller values for *r* to substitute into Theorem 19. We leave that for future work.

Given that the expansion of $\frac{p}{q}$ is $[a_0, a_1, ..., a_n]$, the following table summarizes the continued fraction expansions of various rational numbers that were discussed in this paper.

Real Number	Parity of N	Continued Fraction Expansion
$\frac{p}{q}+\frac{1}{(k+1)q^2}$	even	$[a_0, a_1, ..., a_{n-1}, a_n, k, 1, a_n - 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}+\frac{1}{(k+1)q^2}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n - 1, 1, k, a_n, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}-\frac{1}{(k+1)q^2}$	even	$[a_0, a_1, ..., a_{n-1}, a_n - 1, 1, k, a_n, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}-\frac{1}{(k+1)q^2}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n, k, 1, a_n - 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}+\frac{1}{q^2}$	even	$[a_0, a_1, ..., a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}+\frac{1}{q^2}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n - 1, a_n + 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}-\frac{1}{q^2}$	even	$[a_0, a_1, ..., a_{n-1}, a_n - 1, a_n + 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}-\frac{1}{q^2}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}+\frac{1}{q(kq+2q_{n-1})}$	even	$[a_0, a_1, ..., a_{n-1}, a_n, k, a_n, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}-\frac{1}{q(kq+2q_{n-1})}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n, k, a_n, a_{n-1}, ..., a_2, a_1]$
$\frac{p}{q}+\frac{q_{n-1}}{q(q^2+q_{n-1}^2)}$	even	$[a_0, a_1, ..., a_{n-1}, a_n, a_n, a_{n-1}, ..., a_1]$
$\frac{p}{q}-\frac{q_{n-1}}{q(q^2+q_{n-1}^2)}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n, a_n, a_{n-1}, ..., a_1]$
$\frac{p}{q}+\frac{q_{n-2}}{qq_{n-1}(q^2+q_{n-2})}$	even	$[a_0, a_1, ..., a_{n-1}, a_n, a_{n-1}, ..., a_1]$
$\frac{p}{q}-\frac{q_{n-2}}{qq_{n-1}(q^2+q_{n-2})}$	odd	$[a_0, a_1, ..., a_{n-1}, a_n, a_{n-1}, ..., a_1]$

Table 4

Chapter 5 - Future Work

In Example 7 we looked at the continued fraction expansions that appear in the partial sums of the series $\frac{p}{q}+\sum_{i=2}^{\infty} q^{-i}$ for specific values of *p* and *q*. From studying this example among others, it became evident that much more research can be done on the patterns found in the expansions of $\frac{p}{q}+\sum_{i=2}^{\infty} q^{-i}$. As mentioned at the end of Example 7 we made several conjectures regarding these expansions. To help us illustrate this, consider the following tables which give the expansion of $\frac{p}{q}+\sum_{k=2}^{i} q^{-k}$ for various values of *p, q,* and *i*.

Expression	Expansion
p/q = 137/89	[1, 1, 1, 5, 1, 6]
i = 2	[1, 1, 1, 5, 1, 5, 7, 1, 5, 2]
i = 3	[1, 1, 1, 5, 1, 5, 8, 2, 2, 3, 1, 2, 8, 2]
i = 4	[1, 1, 1, 5, 1, 5, 8, 2, 2, 2, 89, 1, 1, 2, 8, 2]
i = 5	[1, 1, 1, 5, 1, 5, 8, 2, 2, 2, 8010, 1, 1, 2, 8, 2]
i = 6	[1, 1, 1, 5, 1, 5, 8, 2, 2, 2, 712979, 1, 1, 2, 8, 2]

Table 5

Expression	Expansion
p/q = 388/93	[4, 5, 1, 4, 3]
i = 2	[4, 5, 1, 4, 4, 2, 4, 1, 5]
i = 3	[4, 5, 1, 4, 4, 2, 9, 1, 8, 5, 6]
i = 4	[4, 5, 1, 4, 4, 2, 9, 1, 845, 5, 6]
i = 5	[4, 5, 1, 4, 4, 2, 9, 1, 78686, 5, 6]
i = 6	[4, 5, 1, 4, 4, 2, 9, 1,7317899, 5, 6]

Table 6

Expression	Expansion
p/q = 73/458	[0, 6, 3, 1, 1, 1, 6]

i = 2	[0, 6, 3, 1, 1, 1, 7, 5, 1, 1, 1, 3, 6]
i = 3	[0, 6, 3, 1, 1, 1, 7, 5, 1, 1, 5, 3, 3, 1, 2, 3, 1, 2, 6]
i = 4	[0, 6, 3, 1, 1, 1, 7, 5, 1, 1, 5, 3, 1835, 1, 2, 3, 1, 2, 6]
i = 5	[0, 6, 3, 1, 1, 1, 7, 5, 1, 1, 5, 3, 840891, 1, 2, 3, 1, 2, 6]
i = 6	[0, 6, 3, 1, 1, 1, 7, 5, 1, 1, 5, 3, 385128539, 1, 2, 3, 1, 2, 6]

Table 7

From Tables 5, 6, and 7 we see that each expansion eventually becomes fixed apart from one partial quotient. If we cut off the continued fraction of each right before the non-fixed partial quotient, then the value of this rational number is what the series converges to. Now let's observe the non-fixed partial quotients given in each table. In Table 5 they are 89, 8010, and 712979. Similar to Example 7 they can be written as:

(16)
$$\begin{aligned} 89 &= 1(89+1)-1 \\ 8010 &= 1(89^2+89+1)-1 \\ 712979 &= 1(89^3+89^2+89+1)-1. \end{aligned}$$

In Tables 6 and 7 the non-fixed partial quotients given are 8, 845, 78686, 7317899 and 3, 1835, 840891, 385128539 respectively. Once again observe that

(17)
$$\begin{aligned} 8 &= 9(1)-1 \\ 845 &= 9(93+1)-1 \\ 78686 &= 9(93^2+93+1)-1 \\ 7317899 &= 9(93^3+93^2+93+1)-1 \end{aligned}$$

and

(18)
$$\begin{aligned} 3 &= 4(1)-1 \\ 1835 &= 4(458+1)-1 \\ 840891 &= 4(458^2+458+1)-1 \\ 385128539 &= 4(458^3+458^2+458+1)-1. \end{aligned}$$

First, notice from (16), (17), and (18) that the non-fixed partial quotients have the form $k\cdot(q^j+q^{j-1}+\dots+q+1)-1$ for some *j*. In each case above, *k* can be found by finding $[\gcd(p-1,q)]^2$. For instance, $\gcd(137,89)^2=1$, $\gcd(388,93)^2=9$ and $\gcd(73,458)^2=4$.

Secondly, observe that the last set of fixed partial quotients is equal to $\frac{q_n}{q'_{n-1}}$ where q'_{n-1} is the denominator corresponding to the n-1$^{\text{th}}$ convergent of $\frac{p-1}{q}$. In other words, it is the same as the expansion of $\frac{p-1}{q}$ in reverse order if we ignore the first partial quotient. In addition, we sometimes have to start with the form of the expansion that ends in a 1. For example, using the rational numbers from Tables 5, 6, and 7 we see that 136/89 = [1, 1, 1, 8, 2, 2] = [1 ,1, 1, 8, 2, 1, 1], 387/93 = [4, 6, 5], and 72/458 = [0, 6, 2, 1, 3, 3] = [0 ,6, 2, 1, 3, 2, 1] respectively. Writing these in reverse order while ignoring the first partial quotient gives [1, 1, 2, 8, 1, 1] = [1, 1, 2, 8, 2], [5, 6], and [1, 2, 3, 1, 2, 6] all of which appear in the tables above. We now give the following conjecture summarizing the information presented above.

<u>Conjecture</u> Suppose *p* and *q* are rational numbers greater than 1 and the series $\frac{p}{q}+\sum_{i=2}^{\infty} q^{-i}$ converges to some rational number $\frac{u}{v}=[a_0, a_1, ..., a_n]$ where $a_n > 1$. Further assume that the expansion of $\frac{p-1}{q}$ is $[b_0, b_1, ..., b_k]$. Then for some index, j, where j is 3 or 4:

$$\frac{p}{q}+\sum_{i=2}^{j} q^{-i} = [a_0, a_1, ..., a_n, \gcd(p-1, q)^2\left(\sum_{i=0}^{j-3} q^i\right)-1, 1, b_k - 1, ... b_2, b_1]$$

or

$$\frac{p}{q}+\sum_{i=2}^{j} q^{-i} = [a_0, a_1, ..., a_n - 1, 1, \gcd(p-1, q)^2\left(\sum_{i=0}^{j-3} q^i\right)-1, b_k, ... b_2, b_1].$$

Also for each integer $m > j$, we have:

$$\frac{p}{q}+\sum_{i=2}^{m} q^{-i} = [a_0, a_1, ..., a_n, \gcd(p-1, q)^2\left(\sum_{i=0}^{m-3} q^i\right)-1, 1, b_k - 1, ... b_2, b_1]$$

or

$$\frac{p}{q}+\sum_{i=2}^{m}q^{-i}=[a_0,a_1,...,a_n-1,1,\gcd(p-1,q)^2\left(\sum_{i=0}^{m-3}q^i\right)-1,b_k,...b_2,b_1].$$

Future work would involve determining if this conjecture is true, and if so, providing a proof. It would also involve exploring other patterns that arise in the expansions of partial sums of series.

At the end of Chapter 4 we gave a result that given the irrational number $x=[a_0,a_1,...,a_{n-1},a_n,a_{n+1},...]$, then the sum of x and the rational number r has an expansion that preserves the first n partial quotients of x if r is small enough. Future work in this area would involve pushing the limits on how large a value of r we can find that still has this property.

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